On Hypersurface Quotient Singularities of Dimension 4

Li Chiang
Institute of Mathematics
Academia Sinica
Taipei , Taiwan
(e-mail:chiangl@gate.sinica.edu.tw)

Shi-shyr Roan
Institute of Mathematics
Academia Sinica
Taipei , Taiwan
(e-mail: maroan@ccvax.sinica.edu.tw)

Abstract

We consider geometrical problems on Gorenstein hypersurface orbifolds of dimension $n \geq 4$ through the theory of Hilbert scheme of group orbits. For a linear special group G acting on $\mathbb{C}^n$, we study the G-Hilbert scheme, $\mathrm{Hilb}^G(\mathbb{C}^n)$, and crepant resolutions of $\mathbb{C}^n/G$ for G=the A-type abelian group $A_r(n)$. For $n = 4$, we obtain the explicit structure of $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$. The crepant resolutions of $\mathbb{C}^4/A_r(4)$ are constructed through their relation with $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$, and the connections between these crepant resolutions are found by the "flop" procedure of 4-folds. We also make some primitive discussion on $\mathrm{Hilb}^G(\mathbb{C}^n)$ for the G= alternating group $\mathfrak{A}_{n+1}$ of degree $n + 1$ with the standard representation on $\mathbb{C}^n$; the detailed structure of $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ is explicitly constructed.

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1 Introduction

The purpose of this paper is to study some geometrical problems of certain Gorenstein hypersurface orbifolds of dimension 4. The main focus will be on the structure of the newly developed concept of Hilbert scheme of group orbits and its connection with crepant resolutions of the orbifold. For a finite subgroup G in $\mathrm{SL}_n(\mathbb{C})$, the G-Hilbert scheme, $\mathrm{Hilb}^G(\mathbb{C}^n)$, was first introduced by Nakamura et al [6, 8, 9, 14]; one primary goal aims to provide a conceptual understanding of crepant resolutions of $\mathbb{C}^n/G$ for $n = 3$, whose solution was previously known by a computational method, relying heavily on Miller-Blichfeldt-Dickson classification of finite groups in $\mathrm{SL}_3(\mathbb{C})$ [12] and the invariant theory of two simple groups, I_{60} (icosahedral group), H_{168} (Klein group) [11] (for the existence of crepant resolutions, see [18] and references therein). For $n = 2$, $\mathrm{Hilb}^G(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$, hence it has the trivial canonical bundle [8, 9, 14]. For $n = 3$, it was expected that $\mathrm{Hilb}^G(\mathbb{C}^3)$ is a crepant resolution of $\mathbb{C}^3/G$. Recently the affirmative answer was obtained, in [7, 15] for the abelian group G by techniques in toric geometry, and in [2] for a general group G by derived category methods bypassing the geometrical analysis of G-Hilbert scheme. With these successful results in dimension 3, a question naturally arises on the possible role of G-Hilbert scheme on crepant resolution problems of orbifolds in dimension $n \geq 4$. For $n \geq 4$, it is a well-known fact that $\mathbb{C}^n/G$ might have no crepant resolutions at all, even for a cyclic group G and $n = 4$, (for a selection of examples, see e.g., [19]). To avoid many such complicated exceptional cases, we will restrict our attention only to those with hypersurface singularities. In this paper, we will address certain problems on two specific types of hypersurface Gorenstein quotient singularity, $\mathbb{C}^n/G$, of dimension n; one is the abelian group $G = A_r(n)$ defined in (8) in the main body of the paper, the other group G is the alternating group $\mathfrak{A}_{n+1}$ of degree $n+1$ acting on $\mathbb{C}^n$ through the standard representation. In the case $G = A_r(n)$, $\mathrm{Hilb}^G(\mathbb{C}^n)$ is a toric variety, hence the methods for toric geometry provide an effectively tool to study its structure. For $n = 4$, we will give a detailed derivation of the smooth toric structure of $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$, and construct the crepant toric resolutions of $\mathbb{C}^4/A_r(4)$ by blowing-down the canonical divisors of $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$; in due course the "flop" of 4-folds naturally arises in the process, (see Theorems 3.1, 3.2, 4.1 in the main text of this paper, whose statements were previously announced in [3]). We would expect the concept appeared in the proof of these theorems will inspire certain clue to other cases, not only the $A_r(n)$-type groups, but also for the non-abelian groups $\mathfrak{A}_{n+1}$ (which are simple groups for $n \geq 4$). The group $\mathfrak{A}_4$ is a solvable group of order 12, also called the ternary trihedral group. The crepant resolution of $\mathbb{C}^3/\mathfrak{A}_4$ was explicitly constructed in [1], and the structure $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ over the origin orbit of $\mathbb{C}^3/\mathfrak{A}_4$ was analyzed in detail in [6]. Through the representation theory of $\mathfrak{A}_4$, we will give the direct verification that $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ is smooth and a crepant resolution of $\mathbb{C}^3/\mathfrak{A}_4$. Though the conclusion is known by the general result in [2] using qualitative arguments, the object of our detailed analysis aims to reveal that there exist certain common features in determining the structures of G-Hilbert schemes for certain abelian and non-abelian groups G by the computational methods, in hope that the approach could possibly be applied to higher dimensional cases. With this in mind, we will in this paper restrict our attention only to the case $\mathfrak{A}_4$, leave possible generalizations, applications or implications to future work.

This paper is organized as follows. In §2, we will summarize the main features of G-Hilbert scheme of group orbits, and results in toric geometry for the need of later discussion. We will also define the group G which we will consider in this paper. The next two sections will be devoted to the discussion of the structure of $\mathrm{Hilb}^G(\mathbb{C}^4)$ and crepant resolutions of $\mathbb{C}^4/G$ for $G = A_r(4)$. For the simpler terminology to express the ideas, also for the description of geometry of flop of 4-folds, we will consider only the case $A_1(4)$ in §3 to discuss the structure of $\mathrm{Hilb}^{A_1(4)}(\mathbb{C}^4)$. The flop relation between crepant resolutions of $\mathbb{C}^4/A_1(4)$ will be examined in detail through $\mathrm{Hilb}^{A_1(4)}(\mathbb{C}^4)$. In §4, we

will derive the solution of the corresponding problems for $G = A_r(4)$ for a general positive integer r, with much more complicated techniques but a method much in tune with the previous section. In §5, we consider the case $G = \mathfrak{A}_{n+1}$ acting on $\mathbb{C}^n$ through the standard representation for $n = 3$. By employing the structure of the fiber in $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ over the origin orbit of $\mathbb{C}^3/\mathfrak{A}_4$ described in [6], we give an explicit construction of the smooth and crepant structure of $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ using finite group representation theory, along a line similar to the previous two sections in a certain sense. Finally we give the conclusion remarks in §6.

Notations. To present our work, we prepare some notations. In this paper, by an orbifold we shall always mean the orbit space of a smooth complex manifold acted on by a finite group. Throughout the paper, G will always denote a finite group unless otherwise stated. We denote

$$\mathrm{Irr}(G) \quad = \quad \{\rho : G \longrightarrow \mathrm{GL}(V_\rho) \ \text{ an irreducible representation of } G\}.$$

The trivial representation of G will be denoted by $\mathbf{1}$. For a G-module W, i.e., a G-linear representation space W, one has the canonical irreducible decomposition: $W = \bigoplus_{\rho \in \mathrm{Irr}(G)} W_\rho$, where W_ρ is a G-submodule of W, isomorphic to $V_\rho \otimes W_\rho^0$ for some trivial G-module W_ρ^0. For an analytic variety X, we shall not distinguish the notions of vector bundle and locally free $\mathcal{O}_X$-sheaf over X.

2 G-Hilbert Scheme, Toric Geometry

In this section, we brief review some basic facts on $\mathrm{Hilb}^G(\mathbb{C}^n)$ (the Hilbert scheme of G-orbits) and toric geometry necessary for later use, then specify the groups G for the discussion of the rest sections of this paper.

First, we will always assume G to be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$. Denote $S_G := \mathbb{C}^n/G$ with the canonical projection, $\pi_G : \mathbb{C}^n \to S_G$, and $o := \pi_G(\vec{0})$. As G acts on $\mathbb{C}^n$ freely outside a finite collection of linear subspaces with codimension ≥ 2, S_G is an orbifold with non-empty singular set $\mathrm{Sing}(S_G)$ of codimension ≥ 2. In fact, the element o is a singular point of S_G. By a variety X birational over S_G, we will always mean a proper birational morphism σ from X to S_G which is biregular between $X \setminus \sigma^{-1}(\mathrm{Sing}(S_G))$ and $S_G \setminus \mathrm{Sing}(S_G)$,

$$\sigma : X \longrightarrow S_G \ . \tag{1}$$

One can form the commutative diagram via the birational morphism σ,

$$\begin{array}{ccc} X \times_{S_G} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ \downarrow \pi & & \downarrow \pi_G \\ X & \xrightarrow{\sigma} & S_G \ . \end{array} \tag{2}$$

Denote $\mathcal{F}_X$ the coherent $\mathcal{O}_X$-sheaf over X obtained by the push-forward of the structure sheaf of $X \times_{S_G} \mathbb{C}^n$, $\mathcal{F}_X := \pi_* \mathcal{O}_{X \times_{S_G} \mathbb{C}^n}$. For two varieties X, X' birational over S_G with the commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & S_G \\ \mu \downarrow & & || \\ X' & \xrightarrow{\sigma'} & S_G \ , \end{array}$$

one has a canonical morphism, $\mu^* \mathcal{F}_{X'} \longrightarrow \mathcal{F}_X$. In particular, the morphism (1) gives rise to the $\mathcal{O}_X$-morphism,

$$\sigma^*(\pi_{G*} \mathcal{O}_{\mathbb{C}^n}) \longrightarrow \mathcal{F}_X \ .$$

Furthermore, all the above morphisms are compatible with the natural G-structure of $\mathcal{F}_X$ induced from the G-action on $\mathbb{C}^n$ via (2). Then $\mathcal{F}_X$ has the canonical G-decomposition of coherent $\mathcal{O}_X$-submodules: $\mathcal{F}_X = \bigoplus_{\rho \in \mathrm{Irr}(G)} (\mathcal{F}_X)_\rho$, where $(\mathcal{F}_X)_\rho$ is the ρ-factor of $\mathcal{F}_X$. The geometrical fibers of $\mathcal{F}_X$ and $(\mathcal{F}_X)_\rho$ over $x \in X$ are defined by $\mathcal{F}_{X,x} = k(x) \bigotimes_{\mathcal{O}_X} \mathcal{F}_X$, $(\mathcal{F}_X)_{\rho,x} = k(x) \bigotimes_{\mathcal{O}_X} (\mathcal{F}_X)_\rho$, where $k(x)(:= \mathcal{O}_{X,x}/\mathcal{M}_x)$ is the residue field at x. Over $X \setminus \sigma^{-1}(\mathrm{Sing}(S_G))$, $\mathcal{F}_X$ is a vector bundle of rank $|G|$ with the regular G-representation on each geometric fiber. Hence $(\mathcal{F}_X)_\rho$ is a vector bundle over $X \setminus \sigma^{-1}(\mathrm{Sing}(S_G))$ with the rank equal to the dimension of V_ρ. For $x \in X$, there exists a G-invariant ideal $I(x)$ in $\mathbb{C}[Z](:= \mathbb{C}[Z_1, \cdots, Z_n])$ such that the following relation holds,

$$\mathcal{F}_{X,x} = k(x) \bigotimes_{\mathcal{O}_{S_G}} \mathcal{O}_{\mathbb{C}^n}(x) \simeq \mathbb{C}[Z]/I(x) \ . \tag{3}$$

We have $(\mathcal{F}_X)_{\rho,x} \simeq (\mathbb{C}[Z]/I(x))_\rho$. The vector spaces $\mathbb{C}[Z]/I(x)$ form a family of finite dimensional G-modules parametrized by $x \in X$. For $x \notin \sigma^{-1}(\mathrm{Sing}(S_G))$, $\mathbb{C}[Z]/I(x)$ is a regular G-module. In particular, for $X = S_G$ in (3) and $s \in S_G$, the G-invariant ideal $I(s)$ of $\mathbb{C}[Z]$ is generated by the G-invariant polynomials vanishing at $\sigma^{-1}(s)$. Let $\widetilde{I}(s)$ be the ideal of $\mathbb{C}[Z]$ consisting of all polynomials vanishing at $\sigma^{-1}(s)$. Then $\widetilde{I}(s)$ is an G-invariant ideal with $\widetilde{I}(s) \supset I(s)$. For $s = o$, we have $\widetilde{I}(o) = \mathbb{C}[Z]_0$ and $I(o) = \mathbb{C}[Z]_0^G \mathbb{C}[Z]$, where the subscript 0 indicates the maximal ideal of polynomials vanishing at the origin. For a variety X birational over S_G via σ in (1), one has the following relations of G-invariant ideals of $\mathbb{C}[Z]$:

$$\widetilde{I}(s) \supset I(x) \supset I(s) \ , \quad x \in X \ , \ s = \sigma(x) \ .$$

For $x \in X$, there exists a direct sum decomposition of $\mathbb{C}[Z]$ as G-modules,

$$\mathbb{C}[Z] = I(x)^\perp \oplus I(x) \ .$$

Here $I(x)^\perp$ is a finite dimensional G-module isomorphic to $\mathbb{C}[Z]/I(x)$. Similarly, we have G-module decompositions for $s = \sigma(x) \in S_G$,

$$\mathbb{C}[Z] = I(s)^\perp \oplus I(s) \ , \quad \mathbb{C}[Z] = \widetilde{I}(s)^\perp \oplus \widetilde{I}(s)$$

so that the relations, $\widetilde{I}(s)^\perp \subset I(x)^\perp \subset I(s)^\perp$, hold. Note that the above finite dimensional G-modules with superscript $\perp$ are not unique in $\mathbb{C}[Z]$ because there is a choice involved, nonetheless we could choose them such that this inclusions are fulfilled. One has the canonical G-decomposition of $I(x)^\perp$: $I(x)^\perp = \bigoplus_{\rho \in \mathrm{Irr}(G)} I(x)^\perp_\rho$, where the factor $I(x)^\perp_\rho$ is isomorphic to a positive finite sum of copies of V_ρ.

Now we consider the varieties X birational over S_G such that $\mathcal{F}_X$ is a vector bundle. Among all such X, there exists a minimal object, called the G-Hilbert scheme in [8, 9, 14, 15],

$$\sigma_{\mathrm{Hilb}} : \mathrm{Hilb}^G(\mathbb{C}^n) \longrightarrow S_G \ . \tag{4}$$

By the definition of $\mathrm{Hilb}^G(\mathbb{C}^n)$, an element (i.e. closed point) p of $\mathrm{Hilb}^G(\mathbb{C}^n)$ is described by a G-invariant ideal $I(= I(p))$ of $\mathbb{C}[Z]$ of colength $|G|$, and the fiber of the vector bundle $\mathcal{F}_{\mathrm{Hilb}^G(\mathbb{C}^n)}$ over p can be identified with the regular G-module $\mathbb{C}[Z]/I(p)$. For simplicity of notations, we shall also make the identification of the element p with its associated ideal I, and write $I \in \mathrm{Hilb}^G(\mathbb{C}^n)$ in what follows if no confusion arises. For any other X, the map (1) can be factored through a birational morphism λ from X onto $\mathrm{Hilb}^G(\mathbb{C}^n)$ via σ_{Hilb},

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & S_G \\ \lambda \downarrow & & || \\ \mathrm{Hilb}^G(\mathbb{C}^n) & \xrightarrow{\sigma_{\mathrm{Hilb}}} & S_G \ . \end{array}$$

In fact, the ideal $I(x)$ of (3) is a colength $|G|$ ideal in $\mathbb{C}[Z]$, by which the map λ is defined. We will denote X_G the normalization of $\mathrm{Hilb}^G(\mathbb{C}^n)$, which is a normal variety over S_G with the birational morphism from X_G onto S_G. As every biregular automorphism of S_G can always be lifted to one of $\mathrm{Hilb}^G(\mathbb{C}^n)$, hence also to X_G, one has the following result.

LEMMA **2.1** *Let $\mathrm{Aut}(S_G)$ be the group of biregular automorphisms of S_G. Then $\mathrm{Hilb}^G(\mathbb{C}^n), X_G$ are $\mathrm{Aut}(S_G)$-varieties over S_G via $\mathrm{Aut}(S_G)$-morphisms. As a consequence, X_G is a toric variety for an abelian group G.*

Now we are going to summarize some basic facts in toric geometry for the later discussion when the group G is abelian, (for details, see e.g., [5, 10, 16]) . In this case, we consider G as a subgroup of the diagonal group T_0 of $\mathrm{GL}_n(\mathbb{C}^n)$ with the identification $T_0 = \mathbb{C}^{*n}$. Regard $\mathbb{C}^n$ as the partial compactification of T_0,

$$G \subset T_0 \subset \mathbb{C}^n \ .$$

Let T be the torus T_0/G and consider S_G $(= \mathbb{C}^n/G)$ as a T-space,

$$T := T_0/G \ , \quad T \subset S_G \ .$$

The combinatorial data of toric varieties are constructed from the lattices of 1-parameter subgroups and characters of tori T, T_0,

$$\begin{array}{rcl} N(:= \mathrm{Hom}(\mathbb{C}^*, T)) & \supset & N_0(:= \mathrm{Hom}(\mathbb{C}^*, T_0)) \ , \\ M(:= \mathrm{Hom}(T, \mathbb{C}^*)) & \subset & M_0(:= \mathrm{Hom}(T_0, \mathbb{C}^*)) \ . \end{array}$$

For convenience, N_0, N will be identified with the following lattices in $\mathbb{R}^n$ in this paper. Denote by $\{e^i\}_{i=1}^n$ the standard basis of $\mathbb{R}^n$, and define the map $\exp : \mathbb{R}^n \longrightarrow T_0$ by $r(= \sum_{i=1}^n r_i e^i) \mapsto \exp(r) := \sum_i e^{2\pi\sqrt{-1} r_i} e^i$. The lattices N, N_0 are given by

$$N_0 = \mathbb{Z}^n (:= \exp^{-1}(1)) \ , \quad N = \exp^{-1}(G) \ ,$$

and we have $G \simeq N/N_0$. The lattice M_0 dual to N_0 is the standard one in the dual space $\mathbb{R}^{n*}$. In what follows, we shall identify M_0 with the group of monomials in variables $Z_1, \ldots, Z_n$ via the correspondence:

$$I = \sum_{s=1}^n i^s e_s \in M_0 \quad \longleftrightarrow \quad Z^I = \prod_{s=1}^n Z_s^{i_s} \ .$$

The dual lattice M of N is the sublattice of M_0, consisting of all G-invariant monomials. Among the varieties X birational over the T-space S_G, we shall consider only those X with a T-structure. It has been known that these toric varieties X are represented by certain combinatorial data in toric geometry. A toric variety over S_G is described by a fan $\Sigma = \{\sigma_\alpha \mid \sigma \in I\}$ with the first quadrant of $\mathbb{R}^n$ as its support, i.e., a rational convex cone decomposition of the first quadrant in $\mathbb{R}^n$. Equivalently, these combinatorial data can also be described by the intersection of the fan and the standard simplex Δ in the first quadrant,

$$\triangle := \{ r \in \mathbb{R}^n | \sum_i r_i = 1, r_j \geq 0 \ \ \forall \ j \} \ . \tag{5}$$

The corresponding data in $\triangle$ are denoted by $\Lambda = \{\triangle_\alpha \mid \alpha \in I\}$ with $\triangle_\alpha := \sigma_\alpha \cap \triangle$. Then Λ is a polytope decomposition of $\triangle$ with vertices in $\triangle \cap \mathbb{Q}^n$. Note that for $\sigma_\alpha = \{\vec{0}\}$, we have $\triangle_\alpha = \emptyset$. Such

Λ will be called a rational polytope decomposition of $\triangle$, and we will denote X_Λ the toric variety corresponding to Λ. If all vertices of Λ are in N, Λ is called an integral polytope decomposition of $\triangle$. For a rational polytope decomposition Λ of $\triangle$, we define $\Lambda(i) := \{\Delta_\alpha \in \Lambda \mid \dim(\Delta_\alpha) = i\}$ for $-1 \leq i \leq n-1$, (here $\dim(\emptyset) := -1$). The T-orbits in X_Λ are parametrized by $\bigsqcup_{i=-1}^{n-1} \Lambda(i)$. In fact, for $\triangle_\alpha \in \Lambda(i)$, there associates a $(n-1-i)$-dimensional T-orbit, which will be denoted by $\mathrm{orb}(\triangle_\alpha)$. A toric divisor in X_Λ is the closure of an $(n-1)$-dimensional orbit, denoted by $D_v = \overline{\mathrm{orb}(v)}$ for $v \in \Lambda(0)$. The canonical sheaf of X_Λ is expressed by the toric divisors (see, e.g. [5, 10, 16]),

$$\omega_{X_\Lambda} = \mathcal{O}_{X_\Lambda}(\sum_{v \in \Lambda(0)} (m_v - 1) D_v) \ , \tag{6}$$

where m_v is the least positive integer with $m_v v \in N$. In particular, X_Λ is crepant, i.e. , $\omega_{X_\Lambda} = \mathcal{O}_{X_\Lambda}$, if and only if Λ is integral. On the other hand, the smoothness of X_Λ is described by the decomposition Λ to be a simplicial one with the multiplicity one property, i.e., for each $\Lambda_\alpha \in \Delta(n-1)$, the elements $m_v v$ for $v \in \Lambda_\alpha \cap \Lambda(0)$ form a $\mathbb{Z}$-basis of N. The following results are known for toric variety over S_G (see e.g. [17]):

(1) The Euler number of X_Λ is given by $\chi(X_\Lambda) = |\Lambda(n-1)|$.

(2) For a rational polytope decomposition Λ of Δ, any two of the following three properties imply the third one:

$$X_\Lambda : \mathrm{non-singular}, \quad \omega_{X_\Lambda} = \mathcal{O}_{X_\Lambda} \ , \quad \chi(X_\Lambda) = |G|. \tag{7}$$

In this paper, we shall consider only two specific series of hypersurface n-orbifold S_G for $n \geq 2$. The first type can be regarded as a generalization of the A-type Klein surface singularity, the group G is defined as follows,

$$A_r(n) := \{ g \in \mathrm{SL}_n(\mathbb{C}) \mid g : \mathrm{diagonal} \ , \ g^{r+1} = 1 \} \ , \quad r \geq 1 \ . \tag{8}$$

The $A_r(n)$-invariant polynomials in $\mathbb{C}[Z](:= \mathbb{C}[Z_1, \ldots, Z_n])$ are generated by monomials, $X := \prod_{i=1}^n Z_i$ and $Y_j := Z_j^{r+1}$ $(j = 1, \ldots, n)$. Thus $S_{A_r(n)}$ is realized as the hypersurface in $\mathbb{C}^{n+1}$,

$$S_{A_r(n)} : \qquad x^{r+1} = \prod_{j=1}^n y_j \ , \qquad (x, y_1, \ldots, y_n) \in \mathbb{C}^{n+1} \ .$$

The ideal $I(o)$ of $\mathbb{C}[Z]$ for the element $o \in S_{A_r(n)}$ is given by $I(o) = \langle Z_1^{r+1}, \ldots, Z_n^{r+1}, Z_1 \cdots Z_n \rangle$, hence

$$I(o)^\perp = \bigoplus \{ \mathbb{C} Z^I \mid I = (i^1, \ldots, i^n) \ , \ \ 0 \leq i^j \leq r \ , \prod_{j=1}^n i^j = 0 \} \ .$$

For a nontrivial character ρ of $A_r(n)$, the dimension of $I(o)_\rho^\perp$ is always greater than one. In fact, one can describe an explicit set of monomial generators of $I(o)_\rho^\perp$. For example, say $I(o)_\rho^\perp$ containing an element Z^I with $I = (i^1, \ldots, i^n), i^1 = 0$ and $i^s \leq i^{s+1}$, then $I(o)_\rho^\perp$ is generated by Z^Ks with $K = (k^1, \ldots, k^n)$ given by

$$k^s = \begin{cases} r + 1 - i^j + i^s, & \text{if } i^s < i^j \ , \\ i^s - i^j, & \text{otherwise} \ , \end{cases} \tag{9}$$

here j runs through 1 to n. Note that some of the above n-tuples K might coincide. In particular for $r = 1$, the dimension of $I(o)_\rho^\perp$ is equal to 2 for $\rho \neq \mathbf{1}$, with a basis consisting of $Z^I, Z^{I'}$ whose indices satisfy the relations, $0 \leq i^s, i^{s'} \leq 1, i^s + i^{s'} = 1$ for $1 \leq s \leq n$.

The second type of group G is the alternating group $\mathfrak{A}_{n+1}$ (of degree $n+1$) acting on $\mathbb{C}^n$ through the standard representation. The representation is induced from the linear action of the symmetric group $\mathfrak{S}_{n+1}$ on $\mathbb{C}^{n+1}$ by permuting the coordinate indices, then restricting on the subspace

$$V = \{(\tilde{z}_1, \ldots, \tilde{z}_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{j=1}^{n+1} \tilde{z}_j = 0\} \ \simeq \ \mathbb{C}^n \quad . \tag{10}$$

We denote $\mathbb{C}[\tilde{Z}](:= \mathbb{C}[\tilde{Z}_1, \cdots, \tilde{Z}_{n+1}])$ the coordinate ring of the affine $(n+1)$-space $\mathbb{C}^{n+1}$, and their elementary symmetric polynomials $\sigma_k := \sum_{1 \leq i_1 < \ldots < i_k \leq n+1} \tilde{Z}_{i_1} \cdots \tilde{Z}_{i_k}$ for $1 \leq k \leq n+1$. The $\mathfrak{A}_{n+1}$-invariant polynomials in $\mathbb{C}[\tilde{Z}]$ are generated by the above σ_ks and $\delta := \prod_{i<j}(\tilde{Z}_i - \tilde{Z}_j)$ with a relation $\delta^2 = \widetilde{F}(\sigma_1, \sigma_2, \ldots, \sigma_{n+1})$ for certain polynomial $\widetilde{F}$. In fact, $\widetilde{F}$ is a (quasi-)homogeneous polynomial of degree $n(n+1)$ with the weights of σ_k and δ equal to $k, \frac{n(n+1)}{2}$ respectively. Denoted by s_k, d the restriction functions of σ_k, δ on V respectively. Then s_1 is the zero function, and $V/\mathfrak{S}_{n+1} = \mathbb{C}^n$ via the coordinates $(s_2, \ldots, s_{n+1})$. The orbifold $S_{\mathfrak{A}_{n+1}} (= V/\mathfrak{A}_{n+1})$ is a double cover of $\mathbb{C}^n$,

$$S_{\mathfrak{A}_{n+1}} \longrightarrow \mathbb{C}^n = V/\mathfrak{S}_{n+1} \ .$$

Then $V/\mathfrak{S}_{n+1}$ can be realized as a hypersurface in $\mathbb{C}^{n+1}$ with the equation,

$$S_{\mathfrak{A}_{n+1}}: \quad d^2 = F_n(s_2, \ldots, s_{n+1}) \ , \quad (d, s_2, \ldots, s_{n+1}) \in \mathbb{C}^{n+1} \ , \tag{11}$$

where $F_n(s_2, \ldots, s_{n+1}) := \widetilde{F}(0, s_2, \ldots, s_{n+1})$. The polynomial $F_n(s_2, \ldots, s_{n+1})$ has a lengthy expression in general. Here we list the polynomial F_n for $n = 3, 4$:

$$\begin{array}{lcl} F_3(s_2, s_3, s_4) & = & -4s_2^3s_3^2 - 27s_3^4 + 16s_2^4s_4 - 128s_2^2s_4^2 + 144s_2s_3^2s_4 + 256s_4^3 \ ; \\ F_4(s_2, s_3, s_4, s_5) & = & -4s_2^3s_3^2s_4^2 - 27s_3^4s_4^2 + 16s_2^4s_4^3 + 144s_2s_3^2s_4^3 - 128s_2^2s_4^4 + 256s_4^5 - 72s_2^4s_3s_4s_5 \\ & & +108s_3^5s_5 - 630s_2s_3^3s_4s_5 - 1600s_3s_4^3s_5 + 560s_2^2s_3s_4^2s_5 + 16s_2^3s_3^3s_5 - 900s_2^3s_4s_5^2 \\ & & +2250s_3^2s_4s_5^2 + 2000s_2s_4^2s_5^2 + 108s_2^5s_5^2 + 825s_2^2s_3^2s_5^2 - 3750s_2s_3s_5^3 + 3125s_5^4. \end{array} \tag{12}$$

3 $\mathbf{A_1(4)}$-Singularity and Flop of 4-folds

We now study the $A_1(n)$-singularity with $n \geq 4$. The set of N-integral elements in Δ are given by

$$\triangle \cap N = \{e^j \mid 1 \leq j \leq n\} \cup \{v^{i,j} \mid 1 \leq i < j \leq n\} \ ,$$

where $v^{i,j} := \frac{1}{2}(e^i + e^j)$ for $i \neq j$. Other than the simplex Δ itself, there is only one integral polytope decomposition of Δ invariant under all permutations of coordinates, and we will denote it by Ξ. $\Xi(n-1)$ consists of $n+1$ elements: $\triangle_i$ $(1 \leq i \leq n)$ and $\diamond$, where $\triangle_i$ is the simplex generated by e^i and $v^{i,j}$ for $j \neq i$, and $\diamond$ is the closure of $\triangle \setminus \bigcup_{i=1}^n \triangle_i$, equivalently $\diamond =$ the convex hull spanned by $v^{i,j}$s for $i \neq j$. The lower dimensional polytopes of Ξ are the faces of those in $\Xi(n-1)$. X_Ξ has the trivial canonical sheaf. For $n = 2, 3$, X_Ξ is a crepant resolution of $S_{A_1(n)}$. For $n = 4$, one has the following result.

Lemma **3.1** *For $n = 4$, the toric variety X_Ξ is smooth except one isolated singularity, which is the 0-dimensional T-orbit corresponding to $\diamond$.*

Proof. In general, for $n \geq 4$, it is easy to see that for each i, the vertices of Δ_i form a $\mathbb{Z}$-basis of N, e.g., say $i = 1$, it follows from $|A_1(n)| = 2^{n-1}$, and $\det(e^1, v^{1,2}, \cdots, v^{1,n}) = \frac{1}{2^{n-1}}$. Hence X_Ξ

is non-singular near the T-orbits associated to simplices in Δ_i. As $\diamond$ is not a simplex, $\mathrm{orb}(\diamond)$ is always a singular point of X_Ξ. For $n=4$, the statement of smoothness of X_Ξ except $\mathrm{orb}(\diamond)$ follows from the fact that for $1 \leq i \leq 4$, the vertices $v^{i,j} (j \neq i)$ of X_Ξ, together with $\frac{1}{2}\sum_{j=1}^4 e^j$, form a N-basis. $\square$

Remark **3.1** *For $n \geq 4$, the following properties hold for 0-dimensional T-orbits of X_Ξ.*

(1) Denote $x_{\Delta_j} := \mathrm{orb}(\Delta_j) \in X_\Xi$ for $1 \leq j \leq n$. The inverse of the matrix spanned by vertices of Δ_j, $(v^{1,j}, \cdots, v^{j-1,j}, e^j, v^{j+1,j}, \cdots, v^{n,j})^{-1}$, gives rise to affine coordinates $(U_1, \ldots, U_n)$ centered at x_{Δ_j} such that $U_i = Z_i^2 \ \ (i \neq j)$, and $U_j = \frac{Z_j}{Z_1 \cdots \check{Z}_j \cdots Z_n}$. Hence $I(x_{\Delta_j}) = \langle Z_j, Z_i^2, i \neq j \rangle + I(o)$, and we have the regular $A_1(n)$-module structure of $\mathbb{C}[Z]/I(x_{\Delta_j})$,

$$\mathbb{C}[Z]/I(x_{\Delta_j}) \simeq \bigoplus \{\mathbb{C}Z^I \mid I = (i_1, \ldots, i_n), \ i_j = 0, i_k = 0, 1 \text{ for } \ k \neq j\} \ . \tag{13}$$

(2) We shall denote $x_\diamond := \mathrm{orb}(\diamond)$ in X_Ξ. The singular structure of $x_\diamond$ is determined by the $A_1(n)$-invariant polynomials corresponding to the M-integral elements in the cone dual to the one generated by $\diamond$ in $N_\mathbb{R}$. So the $A_1(n)$-invariant polynomials are generated by $X_j := Z_j^2$ and $Y_j := \frac{Z_1 \cdots \check{Z}_j \cdots Z_n}{Z_j}$. Hence $I(x_\diamond) = \langle Z_1 \cdots \check{Z}_j \cdots Z_n \rangle_{1 \leq j \leq n} + I(o)$. Note that for $n=3$, the Y_js indeed form the minimal generators for the invariant polynomials, which implies the smoothness of X_Ξ. For $n \geq 4$, $x_\diamond$ is a singularity, not of the hypersurface type. For $n=4$, the X_j, Y_j $(1 \leq j \leq 4)$ form a minimal set of generators of invariant polynomials, hence the structure near $x_\diamond$ in X_Ξ is the 4-dimensional affine variety in $\mathbb{C}^8$ defined by the relations:

$$x_i y_i = x_j y_j, \quad x_i x_j = y_{i'} y_{j'} \ , \ (x_i, y_i)_{1 \leq i \leq 4} \in \mathbb{C}^8, \tag{14}$$

where $i \neq j$ with $\{i', j'\}$ the complementary pair of $\{i, j\}$.

For the rest of this section, we shall consider only the case $n=4$. We are going to discuss the structure of $\mathrm{Hilb}^{A_1(4)}(\mathbb{C}^4)$ and its connection with crepant resolutions of $S_{A_1(4)}$. The simplex Δ is a tetrahedron, and $\diamond$ is an octahedron; both are acted on by the symmetric group $\mathfrak{S}_4$. The dual polygon of $\diamond$ is the cube. The facets of the octahedron $\diamond$ are labeled by F_j, F_j' for $1 \leq j \leq 4$, where $F_j = \diamond \cap \triangle_j$ and $F_j' = \{\sum_{i=1}^4 x_i e^i \in \diamond \mid x_j = 0\}$. The dual of F_j, F_j' in the cube are vertex, denoted by α_j, α_j' as in Fig. 1.

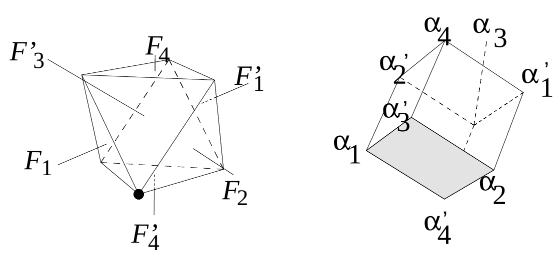

Figure 1: Dual pair of octahedron and cube: Faces F_j, F_j' of octahedron dual to vertices α_j, α_j' of cube. The face of the cube in gray color corresponds to the dot "•" in the octahedron.

Consider the rational simplicial decomposition Ξ^* of Δ, which is a refinement of Ξ by adding the center $c := \frac{1}{4}\sum_{j=1}^4 e^j$ as a vertex with the barycentric decomposition of $\diamond$ in Ξ, (see Fig. 2). Note that $c \notin N$ and $2c \in N$. For convenience, we shall use the following convention:
Notation. Let G be a diagonal group acting on $\mathbb{C}[Z]$. Two monomials m_1, m_2 in $\mathbb{C}[Z]$ are said to be G-equivalent, denoted by $m_1 \overset{G}{\sim} m_2$ or simply by $m_1 \sim m_2$, if m_1/m_2 is a G-invariant function.

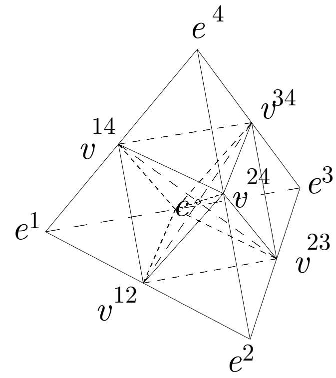

Figure 2: The rational simplicial decomposition Ξ^* of Δ for $n=4, r=1$.

THEOREM **3.1** *For $G = A_1(4)$, we have $\mathrm{Hilb}^G(\mathbb{C}^4) \simeq X_{\Xi^*}$, which is non-singular with the canonical bundle $\omega = \mathcal{O}_{X_{\Xi^*}}(E)$, where E is an irreducible divisor isomorphic to the triple product of $\mathbb{P}^1$,*

$$E = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \ . \tag{15}$$

Furthermore for $\{i,j,k\} = \{1,2,3\}$, the normal bundle of E when restricted on the fiber $\mathbb{P}^1_k$ $(\simeq \mathbb{P}^1)$, for the projection E to $\mathbb{P}^1 \times \mathbb{P}^1$ via the (i,j)-th factor,

$$p_k : E \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \ , \tag{16}$$

is the (-1)-hyperplane bundle:

$$\mathcal{O}_{X_{\Xi^*}}(E) \otimes \mathcal{O}_{\mathbb{P}^1_k} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \ . \tag{17}$$

Proof. First we show the smoothness of the toric variety X_{Ξ^*}. The octahedron $\diamond$ of Ξ is decomposed into eight simplices of Ξ^* corresponding to faces F_j, F'_j of $\diamond$. Denote C_j (resp. C'_j) the simplex of Ξ^* spanned by c and F_j (resp. F'_j); $x_{C_j}, x_{C'_j}$ are the corresponding 0-dimensional T-orbits in X_{Ξ^*}. The smoothness of affine space in X_{Ξ^*} near $x_{C_j}, x_{C'_j}$ follows from the N-integral criterion of the cones in $N_{\mathbb{R}}$ generated by C_j, C'_j. The coordinate system is given by the integral basis of M which generates the cone dual to the cone spanned by C_j (C'_j). As examples, for C_1, C'_2, the coordinates are determined by the row vectors of the following square matrix:

$$\begin{array}{ll} \mathrm{cone}(C_1)^*, & \mathrm{cone}(C'_2)^* \\ (2c, v^{1,2}, v^{1,3}, v^{1,4})^{-1} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} , & (v^{3,4}, 2c, v^{1,4}, v^{1,3})^{-1} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} . \end{array}$$

The coordinate functions of X_{Ξ^*} centered at x_{C_1} are given by $(U_1, U_2, U_3, U_4) = (\frac{Z_2Z_3Z_4}{Z_1}, \frac{Z_1Z_2}{Z_3Z_4}, \frac{Z_1Z_3}{Z_2Z_4}, \frac{Z_1Z_4}{Z_2Z_3})$ with $I(x_{C_1}) = \langle Z_2Z_3Z_4, Z_1Z_2, Z_1Z_3, Z_1Z_4\rangle + I(o)$, and the coordinates near $x_{C'_2}$ are $(U'_1, U'_2, U'_3, U'_4) = (\frac{Z_3Z_4}{Z_1Z_2}, Z_2^2, \frac{Z_1Z_4}{Z_2Z_3}, \frac{Z_1Z_3}{Z_2Z_4})$ with $I(x_{C'_2}) = \langle Z_3Z_4, Z_2^2, Z_1Z_4, Z_1Z_3\rangle + I(o)$. By the Remark 3.1 (1), one has the smooth coordinate system centered at x_{Δ_j} in X_{Ξ^*}. For Δ_1, by

$$\mathrm{cone}(\Delta_1)^*, \quad (e^1, v^{1,2}, v^{1,3}, v^{1,4})^{-1} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} ,$$

one has the coordinate system near x_{Δ_1}, $(V_1, V_2, V_3, V_4) = (\frac{Z_1}{Z_2Z_3Z_4}, Z_2^2, Z_3^2, Z_4^2)$ with $I(x_{\Delta_1}) = \langle Z_1, Z_2^2, Z_3^2, Z_4^2\rangle + I(o)$. Now we are going to show that $\mathbb{C}[Z]/I(y)$ is a regular G-module for $y \in X_{\Xi^*}$. For an element y in the affine neighborhood of x_{Δ_1} with the coordinates $V_i = v_i, (1 \leq i \leq 4)$, one has

$$I(y) = \langle Z_1 - v_1Z_2Z_3Z_4, Z_2^2 - v_2, Z_3^2 - v_3, Z_4^2 - v_4\rangle \tag{18}$$

The set of monomials, $\{1, Z_2, Z_3, Z_4, Z_2Z_3, Z_2Z_4, Z_3Z_4, Z_2Z_3Z_4\}$, gives rise to a basis of $\mathbb{C}[Z]/I(y)$ for $v_i \in \mathbb{C}$; hence $\mathbb{C}[Z]/I(y)$ is a regular G-module. For y near x_{C_1} with the coordinates $U_i = u_i, (1 \leq i \leq 4)$, we have

$$I(y) = \langle Z_2Z_3Z_4 - u_1Z_1, Z_1Z_2 - u_2Z_3Z_4, Z_1Z_3 - u_3Z_2Z_4, Z_1Z_4 - u_4Z_2Z_3\rangle + I_G(y) \ , \tag{19}$$

where $I_G(y) = \langle Z_1Z_2Z_3Z_4 - u_1^2u_2u_3u_4, Z_1^2 - u_1u_2u_3u_4, Z_2^2 - u_2u_1, Z_3^2 - u_3u_1, Z_4^2 - u_4u_1\rangle$. This implies that $\mathbb{C}[Z]/I(y)$ is a regular G-module with a basis represented by $\{1, Z_1, Z_2, Z_3, Z_4, Z_2Z_3, Z_3Z_4, Z_2Z_4\}$. Similarly, the same conclusion holds for y near $x_{C_2'}$ with the coordinates $U_i' = u_i', (1 \leq i \leq 4)$, in which case we have

$$I(y) = \langle Z_3Z_4 - u_1'Z_1Z_2, Z_1Z_4 - u_3'Z_2Z_3, Z_1Z_3 - u_4'Z_2Z_4\rangle + I_G(y) \ , \tag{20}$$

with $I_G(y) = \langle Z_1Z_2Z_3Z_4 - {u_2'}^2u_1'u_3'u_4', Z_2^2 - u_2', Z_1^2 - u_2'u_3'u_4', Z_3^2 - u_1'u_2'u_4', Z_4^2 - u_1'u_2'u_3'\rangle$, and a basis of $\mathbb{C}[Z]/I(y)$ represented by $\{1, Z_1, Z_2, Z_3, Z_4, Z_1Z_2, Z_2Z_3, Z_2Z_4\}$. The same argument can equally be applied to all affine charts centered at $x_{\Delta_j}, x_{C_j}, x_{C_j'}$. Therefore we obtain a morphism

$$\lambda : X_{\Xi^*} \longrightarrow \mathrm{Hilb}^G(\mathbb{C}^4) \ , \quad \text{with} \ \ I(\lambda(y)) = I(y), \ y \in X_{\Xi^*} \ .$$

We are going to show that the above morphism λ is an isomorphism by constructing its inverse morphism. Let y' be an element of $\mathrm{Hilb}^G(\mathbb{C}^4)$, represented by a G-invariant ideal $J \subset \mathbb{C}[Z]$ with $\mathbb{C}[Z]/J$ as the regular G-module. By Gröbner basis techniques [4], for a given monomial order, there is a monomial ideal $\mathrm{lt}(J)$, consisting of all leading monomials of elements in J, such that the monomial base of $\mathbb{C}[Z]/\mathrm{lt}(J)$ also gives rise to a basis of $\mathbb{C}[Z]/J$. By this fact, we shall first determine the G-invariant monomial ideal J_0 in $\mathrm{Hilb}^G(\mathbb{C}^4)$. For a monomial I, we shall denote $I^\dagger$ the set of monic monomials not in I. Since all nonconstant G-invariant monomials are in J_0, we have ${Z_j}^2, Z_1Z_2Z_3Z_4 \in J_0$. Hence $J_0^\dagger$ is contained in the set $\mathcal{B} := \{Z^I \mid I = (i_1, .., i_4), \ i_1i_2i_3i_4 = 0, i_j \leq 1\}$. For a nontrivial character ρ of G, the ρ-eigenspace of $I(o)^\perp$ for the element $o \in S_G$ is of dimension 2. This implies that for $m_1 \in \mathcal{B}$ not equal to 1, there exists exactly one $m_2 \in \mathcal{B}$ not equal to m_1 with $m_2 \sim m_1$. When $J_0 = I(x_{\Delta_1})$, $I(x_{\Delta_1})^\perp$ has a monomial basis $W := I(x_{\Delta_1})^\dagger$ consisting of eight elements ${Z_2}^{i_2}{Z_3}^{i_3}{Z_4}^{i_4}$, $0 \leq i_j \leq 1$, and they form a basis of the G-regular representation. By replacing some monomials in W by the other G-equivalent ones in $\mathcal{B}$, one obtains a G-regular basis W'. Denote W_0 the set of monic monomials in $\mathbb{C}[Z]$. The W's satisfying $W_0 \cdot (W_0 - W') \subset (W_0 - W')$ are in one-to-one correspondence with monomial ideals J_0s in $\mathrm{Hilb}^G(\mathbb{C}^4)$ by the relation $J_0 = \langle W_0 - W'\rangle_{\mathbb{C}}$, hence $W' = J_0^\dagger$. By direct counting, there are twelve such W' and the corresponding twelve J_0's, are exactly those $I(x_{\mathfrak{R}})$ for $\mathfrak{R} \in \Xi^*(3)$. The correspondence

between W' and $\mathfrak{R}$ by the relation $W' = I(x_{\mathfrak{R}})^\dagger$ is given as follows:

$$
\begin{array}{ccc}
\{1, Z_2, Z_3, Z_4, Z_2Z_3, Z_2Z_4, Z_3Z_4, Z_2Z_3Z_4\} & \leftrightarrow & \Delta_1, \\
\{1, Z_1, Z_3, Z_4, Z_1Z_4, Z_1Z_3, Z_3Z_4, Z_1Z_3Z_4\} & \leftrightarrow & \Delta_2, \\
\{1, Z_1, Z_2, Z_4, Z_1Z_4, Z_2Z_4, Z_1Z_2, Z_1Z_2Z_4\} & \leftrightarrow & \Delta_3, \\
\{1, Z_1, Z_2, Z_3, Z_2Z_3, Z_1Z_3, Z_1Z_2, Z_1Z_2Z_3\} & \leftrightarrow & \Delta_4, \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_2Z_3, Z_2Z_4, Z_3Z_4\} & \leftrightarrow & C_1, \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_1Z_4, Z_1Z_3, Z_3Z_4\} & \leftrightarrow & C_2, \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_1Z_4, Z_2Z_4, Z_1Z_2\} & \leftrightarrow & C_3, \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_2Z_3, Z_1Z_3, Z_1Z_2\} & \leftrightarrow & C_4, \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_1Z_4, Z_1Z_3, Z_1Z_2\} & \leftrightarrow & C_1', \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_2Z_3, Z_2Z_4, Z_1Z_2\} & \leftrightarrow & C_2', \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_2Z_3, Z_1Z_3, Z_3Z_4\} & \leftrightarrow & C_3', \\
\{1, Z_1, Z_2, Z_3, Z_4, Z_1Z_4, Z_2Z_4, Z_3Z_4\} & \leftrightarrow & C_4'.
\end{array}
\tag{21}
$$

Now we consider an ideal J in $\mathbb{C}[Z]$ which defines an element of $\mathrm{Hilb}^G(\mathbb{C}^4)$. By the Gröbner basis argument as before, there is a monomial ideal $J_0 (= \mathrm{lt}(J))$ such that $J_0^\dagger$ gives rise to a basis of $\mathbb{C}[Z]/J$, and $J_0 = I(x_{\mathfrak{R}})$ for some $\mathfrak{R} \in \Xi^*(3)$. For $p \in \mathbb{C}[Z]$, the element $p + J \in \mathbb{C}[Z]/J$ is uniquely expressed in the form, $p + J = \sum_{m \in J_0^\dagger} \gamma(p)_m m + J$, i.e., $p - \sum_{m \in J_0^\dagger} \gamma(p)_m m \in J$. In particular, for a monomial p in $\mathbb{C}[Z]$, (i.e., $p \in W_0$), we have $g \cdot (p - \sum_{m \in J_0^\dagger} \gamma(p)_m m) \in J$ for $g \in G$. This implies $p - \sum_{m \in J_0^\dagger} \gamma(p)_m \mu_g(p)^{-1} \mu_g(m) m \in J$, where $\mu_g(m), \mu_g(p) \in \mathbb{C}^*$ are the the character values of g on m, p respectively; hence

$$
\sum_{m \in J_0^\dagger} \gamma(p)_m \left[\mu_g(p)^{-1} \mu_g(m) - 1 \right] m \in J.
$$

As $J_0^\dagger$ represents a G-regular basis for $\mathbb{C}[Z]/J$, we have $\gamma(p)_m \left[\mu_g(p)^{-1} \mu_g(m) - 1 \right] = 0$ for $p \in W_0$, $m \in J_0^\dagger$ and $g \in G$. Furthermore, for each $p \in W_0$, there exists an unique element, denoted by $p_{J_0^\dagger}$, in $J_0^\dagger$ with the property $p \sim p_{J_0^\dagger}$. Hence for $m \in J_0^\dagger$, $m \neq p_{J_0^\dagger}$ if and only if $\left[\mu_g(p)^{-1} \mu_g(m) - 1 \right] \neq 0$ for some $g \in G$, in which case $\gamma(p)_m = 0$. Therefore $p - \gamma(p)_{p_{J_0^\dagger}} p_{J_0^\dagger} \in J$, and J is the ideal with the generators:

$$
J = \langle \; p - \gamma(p)_{p_{J_0^\dagger}} p_{J_0^\dagger} \mid p \in W_0 \cap J_0 \; \rangle \; . \tag{22}
$$

Indeed in the above expression of J, it suffices to consider those ps which from a minimal set of monomial generators of J_0. Now we are going to assign an element of X_{Ξ^*} for a given $J \in \mathrm{Hilb}^G(\mathbb{C}^4)$. If the monomial ideal J_0 associated to J in our previous discussion is equal to $I(x_{C_1})$, a minimal set of monomial generators of J_0 and the basis representative set $J_0^\dagger$ of $\mathbb{C}[Z]/J$ are given by

$$
J_0 = \langle Z_1^2, Z_2^2, .., Z_4^2, Z_1Z_2, Z_1Z_3, Z_1Z_4, Z_2Z_3Z_4 \rangle,
$$

$$
J_0^\dagger = \{1, Z_1, Z_2, Z_3, Z_4, Z_2Z_3, Z_2Z_4, Z_3Z_4\}.
$$

By (22), J contains the ideal generated by $p - \gamma(p)_{p_{J_0^\dagger}} p_{J_0^\dagger}$ for $p = Z_i^2, Z_1Z_2, Z_1Z_3, Z_1Z_4, Z_2Z_3Z_4$ for $1 \leq i \leq 4$, which has the colength at most 8 in $\mathbb{C}[Z]$. Therefore one obtains

$$
\begin{array}{rl}
J = & \langle Z_1Z_4 - \gamma_{14} Z_2Z_3, \;\; Z_1Z_3 - \gamma_{13} Z_2Z_4, \;\; Z_1Z_2 - \gamma_{12} Z_3Z_4, Z_2Z_3Z_4 - \gamma_{234} Z_1, \\
& {Z_1}^2 - \gamma_1, \;\; {Z_2}^2 - \gamma_2, \;\; {Z_3}^2 - \gamma_3, \;\; {Z_4}^2 - \gamma_4 \; \rangle
\end{array}
$$

Moreover, by

$$0 \equiv Z_2({Z_1}^2 - \gamma_1) - Z_1(Z_1Z_2 - \gamma_{12}Z_3Z_4) \equiv (\gamma_{12}\gamma_{13}\gamma_4 - \gamma_1)Z_2 \ (\mathrm{mod} J),$$

and $Z_2 \in {J_0}^\dagger$, one has

$$\gamma_1 = \gamma_{12}\gamma_{13}\gamma_4 \quad .$$

By

$$0 \equiv Z_1({Z_4}^2 - \gamma_4) - Z_4(Z_1Z_4 - \gamma_{14}Z_2Z_3) \equiv (\gamma_{14}\gamma_{234} - \gamma_4)Z_1 \ (\mathrm{mod} J),$$

one obtains

$$\gamma_2 = \gamma_{234}\gamma_{12}.$$

Similarly, one has

$$\gamma_3 = \gamma_{234}\gamma_{13}, \gamma_4 = \gamma_{234}\gamma_{14},$$

Therefore, all γ_Is are expressed as functions of $\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{234}$. This implies $J = I(y)$ for an element y of X_{Ξ^*} in the affine neighborhood x_{C_1} with the coordinate $(U_i = u_i)$ by the relations,

$$u_1 = \gamma_{234}, \quad u_2 = \gamma_{12}, \quad u_3 = \gamma_{13}, \quad u_4 = \gamma_{14}.$$

The above y is defined to be the element $\lambda^{-1}(J)$ in X_{Ξ^*} for the ideal J under the inverse map of λ. The method can equally be applied to ideals J associated to another monomial ideal J_0.

For $J_0 = I(x_{C_2'})$, we have

$$J = \langle Z_1Z_3 - \gamma'_{13}Z_2Z_4,\ Z_1Z_4 - \gamma'_{14}Z_2Z_3,\ Z_3Z_4 - \gamma'_{34}Z_1Z_2,\ {Z_1}^2 - \gamma'_1,\ {Z_2}^2 - \gamma'_2,\ {Z_3}^2 - \gamma'_3,\ {Z_4}^2 - \gamma'_4\rangle.$$

We claim that the variables γ'_2, γ'_{34}, γ'_{13}, γ'_{14} form a system of coordinates near $I(x_{C_2'})$, i.e., all the γ'_Is can be expressed as certain polynomials of these four values. Indeed, we are going to show $\gamma'_1 = \gamma'_2\gamma'_{13}\gamma'_{14}$, $\gamma'_3 = \gamma'_2\gamma'_{13}\gamma'_{34}$ and $\gamma'_4 = \gamma'_2\gamma'_{14}\gamma'_{34}$.[1] By

$$Z_1(Z_1Z_4 - \gamma'_{14}Z_2Z_3) - Z_4({Z_1}^2 - \gamma'_1) = -\gamma'_{14}Z_1Z_2Z_3 + \gamma'_1Z_4 \in J,$$

one has

$$Z_2(-\gamma'_{14}Z_1Z_2Z_3 + \gamma'_1Z_4) + \gamma'_{14}Z_1Z_3(Z_2^2 - \gamma'_2) = \gamma'_1Z_2Z_4 - \gamma'_{14}\gamma'_2Z_1Z_3 \in J,$$

hence

$$(\gamma'_1Z_2Z_4 - \gamma'_{14}\gamma'_2Z_1Z_3) + \gamma'_{14}\gamma'_2(Z_1Z_3 - \gamma'_{13}Z_2Z_4) = (\gamma'_1 - \gamma'_2\gamma'_{13}\gamma'_{14})Z_2Z_4 \in J.$$

By the description in (21) for C'_2, Z_2Z_4 is an element in $J_0^\dagger$, hence represents a basis element of $\mathbb{C}[Z]/J$. The relation $(\gamma'_1 - \gamma'_2\gamma'_{13}\gamma'_{14})Z_2Z_4 \in J$ implies

$$\gamma'_1 - \gamma'_2\gamma'_{13}\gamma'_{14} = 0 \ .$$

By interchanging the indices 1 and 3, (resp. 1 and 4), in the above derivation and regarding $\gamma'_{ij} = \gamma'_{ji}$, we obtain $\gamma'_3 = \gamma'_2\gamma'_{13}\gamma'_{34}$ (resp. $\gamma'_4 = \gamma'_2\gamma'_{14}\gamma'_{34}$). Thus, γ'_2, γ'_{13}, γ'_{13} and γ'_{34} form the

[1] Note that the group G in Section 6.1 of [15] (page 777) is the $A_1(4)$ of Theorem 3.1 in this paper. However, we would consider that the statement in [15] about the singular property of $\mathrm{Hilb}^G(\mathbb{C}^4)$ by using the structure of $I(\Gamma_3)(u)$ there, is not correct. Indeed, by identifying Z_2, Z_3, Z_4, Z_1 with x, y, z, w, and $\gamma'_2, \gamma'_3, \gamma'_4, \gamma'_1, \gamma'_{34}, \gamma'_{13}, \gamma'_{14}$ with $u_1, u_2, \cdots, u_7$ respectively, the ideal J in our discussion corresponds to $I(\Gamma_3)(u)$ in [15]. Then through the three relations we have obtained here, one can easily verify that all the relations among the u_is listed in page 778 of [15] hold.

four independent parameters to describe the ideals J near $J_0 = I(x_{C_2'})$ with the regular G-module $\mathbb{C}[Z]/J$. Therefore $J = I(y)$ for y near $x_{C_2'}$ with the coordinates $(U_i' = u_i')$ via the relations,

$$u_2' = \gamma_2', \quad u_1' = \gamma_{34}', \quad u_3' = \gamma_{14}', \quad u_4' = \gamma_{13}'.$$

For $J_0 = I(x_{\Delta_1})$, we have $J = \langle Z_1 - \gamma_1'' Z_2Z_3Z_4, {Z_2}^2 - \gamma_2'', {Z_3}^2 - \gamma_3'', {Z_4}^2 - \gamma_4'' \rangle$. Hence $J = I(y)$ for y near x_{Δ_1} with the coordinates $(V_i = v_i)$ and the relations, $v_i = \gamma_i''$ for $1 \leq i \leq 4$. The previous discussions of three cases can be applied to each of the twelve monomial ideals J_0's by a suitable change of indices. Hence one obtains an element $\lambda^{-1}(J)$ in X_{Ξ^*} of an ideal $J \in \mathrm{Hilb}^G(\mathbb{C}^4)$.

However, one has to verify the correspondence λ^{-1} so defined to be a single-valued map, namely, for a given J with two possible choices of J_0, the elements in X_{Ξ^*} assigned to J through the previous procedure through these two J_0 are the same one. For example, say $J = I(y_1) = I(y_2)$ for y_1 near x_{Δ_1} with $(V_i = v_i)$, and y_2 near x_{C_1} with $(U_i = u_i)$. By (18), (19), both $Z_2Z_3Z_4 - u_1Z_1$ and $Z_1 - v_1Z_2Z_3Z_4$ are elements in J. We claim that $u_1 \neq 0$. Otherwise, both Z_1 and $Z_2Z_3Z_4$ are elements in J with the same G-character κ. Then the κ-eigenspace in $\mathbb{C}[Z]/J$ is the zero space, a contradiction to the regular G-module property of $\mathbb{C}[Z]/J$. Hence one has $Z_1 - {u_1}^{-1}Z_2Z_3Z_4 \in J$, hence $(v_1 - {u_1}^{-1})Z_2Z_3Z_4 \in J$. As $J = I(y_1)$ with y_1 near x_{Δ_1}, $Z_2Z_3Z_4$ represents a basis element of $\mathbb{C}[Z]/J$. Hence $v_1 = {u_1}^{-1}$. By $Z_1Z_2 - u_2Z_3Z_4,\ {Z_2}^2 - v_2 \in J$, one has $v_2Z_1 - u_2Z_2Z_3Z_4 (= (Z_1Z_2 - u_2Z_3Z_4)Z_2 - ({Z_2}^2 - v_2)Z_1) \in J$. As $Z_2Z_3Z_4 \notin J$, one has $u_2 = 0$ if $v_2 = 0$. When $v_2 \neq 0$, we have, $Z_1 - u_2{v_2}^{-1}Z_2Z_3Z_4 \in J$, hence

$$(v_1 - u_2{v_2}^{-1})Z_2Z_3Z_4 \in J, \quad u_2 = v_1v_2 \ .$$

Using the same argument, one can derives $u_j = v_1v_j$ for $j = 2, 3, 4$. These three relations, together with $u_1 = {v_1}^{-1}$, imply $y_1 = y_2$ in X_{Ξ^*}.

For y_2 near x_{C_1} with $(U_i = u_i)$, and y_3 near $x_{C_2'}$ with $(U_i' = u_i')$, by (19) (20), both $Z_1Z_2 - u_2Z_3Z_4$ and $Z_3Z_4 - u_1'Z_1Z_2$ are elements in J; furthermore, u_2, u_1' are non-zero by the fact that only one of Z_1Z_2, Z_3Z_4 could be an element of J. By an argument similar to the one before, one can show

$$u_1' = {u_2}^{-1}, \quad u_3 = u_4', \quad u_4 = u_3'.$$

By $Z_2Z_3Z_4 - u_1Z_1,\ Z_3Z_4 - u_1'Z_1Z_2,\ {Z_2}^2 - u_2' \in J$, we have

$$(Z_2Z_3Z_4 - u_1Z_1)Z_2 \equiv (u_1'u_2' - u_1)Z_1Z_2 \equiv 0 \mod J \ .$$

As Z_1Z_2 represents a basis element of $\mathbb{C}[Z]/J$, one has $u_1 = u_1'u_2'$. The four relations between u_is and u_i's imply $y_2 = y_3$ in X_{Ξ^*}. In this way, one can show directly that for a given ideal J with $J = I(y) = I(y')$ for y, y' in X_{Ξ^*}, the elements y and y' are the same one by the relations of toric coordinates centered at two distinct $x_{\mathfrak{R}}$s. Hence we have obtained a well-defined morphism λ^{-1} from $\mathrm{Hilb}^G(\mathbb{C}^4)$ to X_{Ξ^*}, then $\mathrm{Hilb}^G(\mathbb{C}^4) \simeq X_{\Xi^*}$. By (6), the canonical bundle of X_{Ξ^*}. is given by $\omega = \mathcal{O}_{X_{\Xi^*}}(E)$, where E denotes the toric divisor D_c, which is a 3-dimensional complete toric variety with the toric data described by the star of c in Ξ^*, which is represented by the octahedron in Fig. 1, where the cube in Fig. 1 represents the toric orbits' structure. Therefore E is isomorphic to the triple product of $\mathbb{P}^1$ as in (15). The description of the normal bundle of E restricting on each $\mathbb{P}^1$-fiber will follow by the direct computation in toric geometry. For example, for the fibers over the projection of E onto $(\mathbb{P}^1)^2$ corresponding to the 2-convex set spanned by $v^{1,2}, v^{1,3}, v^{3,4}$ and $v^{2,4}$, one can perform the computation as follows. Let (U_1, U_2, U_3, U_4) be the local coordinates near $x_{C_4'}$ dual to the N-basis $(2c, v^{1,2}, v^{1,3}, v^{2,3})$, and let (W_1, W_2, W_3, W_4) be the local coordinates near x_{C_1} dual to $(2c, v^{1,2}, v^{1,3}, v^{1,4})$. By $2c = v^{1,4} + v^{2,3}$, one has the relations, $U_1 = W_1W_4$, $U_4 = W_4^{-1}$,

$U_2 = W_2$, $U_3 = W_3$. This shows that the restriction of the normal bundle of E on each fiber $\mathbb{P}^1$ over (U_2, U_3)-plane is the (-1)-hyperplane bundle. $\Box$

Note that the vector bundle $\mathcal{F}_{X_{\Xi^*}}$ over X_{Ξ^*} in Theorem 3.1 carries the regular G-module structure on each fiber with the local frame of the vector bundle provided by the structure of $\mathbb{C}[Z]/I(x_{\mathfrak{R}})$ for $\mathfrak{R} \in \Xi^*(3)$ with the representative in the list (21).

By the standard blowing-down criterion of an exceptional divisor, the property (17) ensures the existence of a smooth 4-fold $(X_{\Xi^*})_k$ by blowing-down the $\mathbb{P}^1$-family along the projection p_k (16) for each k. In fact, $(X_{\Xi^*})_k$ is also a toric variety X_{Ξ_k} with Ξ_k defined by the refinement of Ξ by adding the segment connecting $v^{k,4}$ and $v^{i,j}$ to divide the central polygon $\diamond$ into four simplices, where $\{i, j, k\} = \{1, 2, 3\}$. Each X_{Ξ_k} is a crepant resolution of $X_{\Xi}(= S_G)$, and one has the refinement relation of toric varieties : $\Xi \prec \Xi_k \prec \Xi^*$ for $k = 1, 2, 3$. The polyhedral decomposition in the central core $\diamond$ appeared in the refinements is indicated by the following relation,

$$\diamond \prec \diamond_k \prec \diamond^* \ , \qquad k = 1, 2, 3,$$

whose pictorial realization is shown in Fig. 3. The connection between these three smooth 4-folds

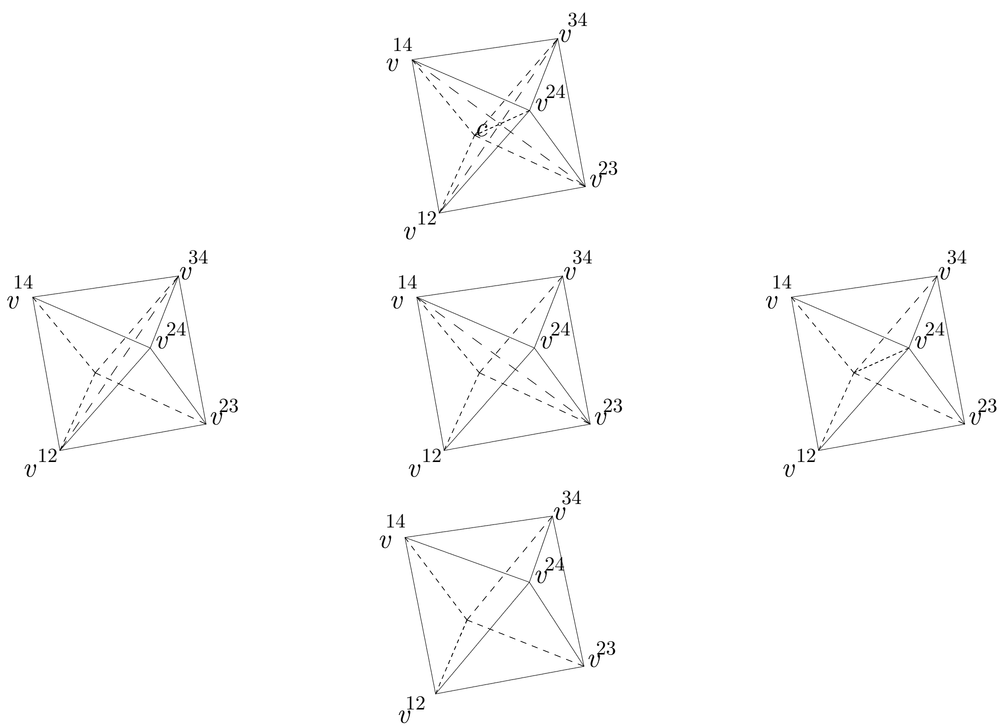

Figure 3: Toric representation of 4-dimensional flops in the second row over a common singular base in the third row and dominated by the same 4-fold in the first row.

corresponding to these different $\diamond_k$s can be regarded as the "flop" relation of 4-folds, an analogy to the similar procedure in birational geometry of 3-folds [13]. Each one is a "small"[2] resolution of the 4-dimensional isolated singularity with the defining equation (14). Hence we have shown the following result.

[2]Here the "smallness" for a resolution means one with the exceptional locus of codimension ≥ 2.

Theorem **3.2** *For $G = A_1(4)$, there are crepant resolutions of S_G obtained by blowing down the divisor E of* $\mathrm{Hilb}^G(\mathbb{C}^4)$ *along* (16) *in Theorem* 3.1. *Any two such resolutions differ by a "flop" of 4-folds.*

4 G-Hilbert Scheme, Crepant Resolution of $\mathbb{C}^4/\mathbf{A_r}(4)$

In this section, we give a complete proof of a general result as in Theorem 3.2, but on the group $A_r(4)$ for all r.

Theorem **4.1** *For $G = A_r(4)$, the G-Hilbert scheme* $\mathrm{Hilb}^G(\mathbb{C}^4)$ *is a non-singular toric variety with the canonical bundle, $\omega = \mathcal{O}_{\mathrm{Hilb}^G(\mathbb{C}^4)}(\sum_{k=1}^m E_k)$ with $m = \frac{r(r+1)(r+2)}{6}$, where E_ks are disjoint smooth exceptional divisors in* $\mathrm{Hilb}^G(\mathbb{C}^4)$*, each of which satisfies the conditions (15) (17). By blowing down E_k to $\mathbb{P}^1 \times \mathbb{P}^1$ via a projection (16) for each k, it gives rise to a toric crepant resolution $\widehat{S}_G$ of S_G with $\chi(\widehat{S}_G) = |A_r(4)| = (r+1)^3$. Furthermore, any two such $\widehat{S}_G$s differ by a sequence of flops.*

Proof. First we define the simplicial decomposition Ξ^* of (5) for $n = 4$, and then we will show that the toric variety X_{Ξ^*} is isomorphic to $\mathrm{Hilb}^G(\mathbb{C}^4)$. We shall denote an element of $N \cap \Delta$ by

$$\mathbf{v}^m (= \mathbf{v}^{(m_1,..,m_4)^t}) := \frac{m_1 e^1 + m_2 e^2 + m_3 e^3 + m_4 e^4}{r+1} \ , \quad 0 \leq m_i \leq r+1 \ , \ \sum_{i=1}^4 m_i = r+1 \ .$$

For each $\mathbf{v}^m \in N \cap \Delta$, there are four hyperplanes passing through $\mathbf{v}^m$, and parallel to one of the four facets of Δ. The collection of all such hyperplanes gives rise to a polytope decomposition of Δ, denoted by Ξ, (for $r = 2$ see the left one of Fig. 4).

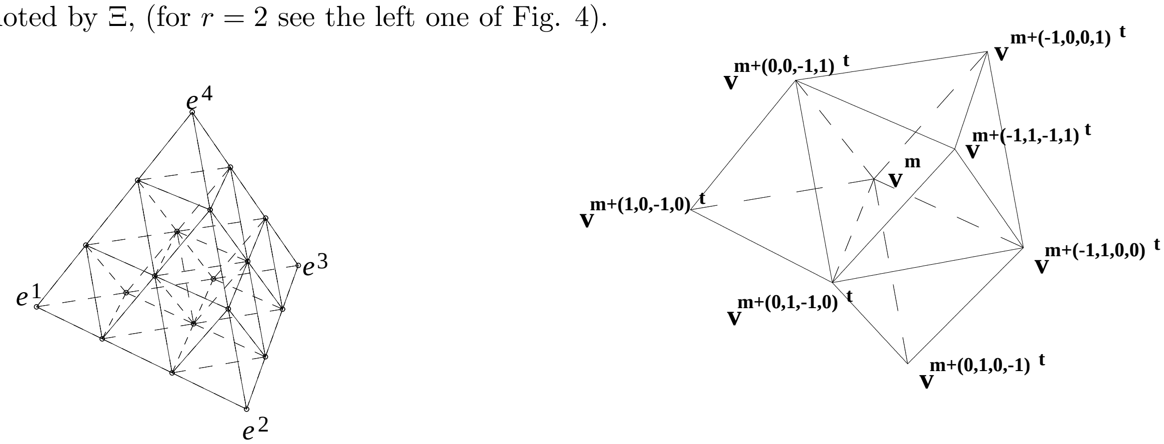

Figure 4: The polytope decomposition Ξ of Δ for $r = 2$ and local figure of Ξ.

Now we examine the polytope structure of Ξ. We have $\Xi(0) = N \cap \Delta$. For each $\mathbf{v}^m \in \Xi(0)$, there are at most twelve segments in $\Xi(1)$ containing $\mathbf{v}^m$, and they are given by $\langle \mathbf{v}^m, \mathbf{v}^{m(i,j)} \rangle$ for $i \neq j$, $1 \leq i,j \leq 4$, where $m(i,j) := m + e^i - e^j$. For a given $\langle \mathbf{v}^m, \mathbf{v}^{m(i,j)} \rangle$, the hyperplane passing $\mathbf{v}^m$ in $\mathbb{R}^4$ with the normal vector $e^i - e^j$ separates Δ into two polytopes Δ's, (one of which could possibly be the empty set). We are going to discuss those elements in Ξ containing $\mathbf{v}^m$ and lying in a non-empty polytope of these two divided ones. For easier description of our conclusion, also for the simplicity of notions, we shall work on a special model case, say $i = 2, j = 3$, and the non-empty polytope Δ' consisting of those elements in Δ with non-negative inner-product to $e^2 - e^3$,

(no difficulties for a similar discussion will arise on other cases except for a suitable change of indices). The elements in $\Xi(3)$ contained in Δ' with $\mathbf{v}^m$ as one of its vertices are the following ones:

$$
\begin{array}{ll}
\Delta_u := \langle \mathbf{v}^m, \mathbf{v}^{m(2,3)}, \mathbf{v}^{m(1,3)}, \mathbf{v}^{m(4,3)} \rangle, & \Delta_d := \langle \mathbf{v}^m, \mathbf{v}^{m(2,3)}, \mathbf{v}^{m(2,1)}, \mathbf{v}^{m(2,4)} \rangle, \\
\diamond_+ := \langle \mathbf{v}^m, \mathbf{v}^{m(2,3)}, \mathbf{v}^{m(4,3)}, \mathbf{v}^{m(2,1)}, \mathbf{v}^{m(4,1)}, \mathbf{v}^{m+(-1,1,-1,1)^t} \rangle, & \\
\diamond_- := \langle \mathbf{v}^m, \mathbf{v}^{m(2,3)}, \mathbf{v}^{m(1,3)}, \mathbf{v}^{m(2,4)}, \mathbf{v}^{m(1,4)}, \mathbf{v}^{m+(1,1,-1,-1)^t} \rangle \ . &
\end{array} \tag{23}
$$

Note that $\diamond_\pm$ are similar by interchanging e^3 and e^4, (for the configuration of $\Delta_u, \Delta_d, \diamond_+$, see the right one of Fig. 4). Both Δ_u, Δ_d are 3-simplices with their vertices forming an integral basis of N, and one facet of each of these 3-simplices is parallel to that of Δ. The toric data of Δ_u, Δ_d give rise to the smooth affine open subsets of X_Ξ. The polytope $\diamond_+$ ($\diamond_-$) is an octahedron with the center $c = \mathbf{v}^m + \frac{e^2+e^4-e^1-e^3}{2(r+1)}$ ($c = \mathbf{v}^m + \frac{e^1+e^2-e^3-e^4}{2(r+1)}$ respectively). We shall mark the octahedron by its center c, and denote it by $\diamond^c$. The affine open subset of X_Ξ with the toric data $\diamond^c$ is smooth except one isolated singular point $x_{\diamond^c}$, an 0-dimensional toric orbit of the affine toric variety. Hence, one can conclude that $\Xi(3)$ consists of three type of elements: Δ_u, Δ_d or $\diamond^c$. The toric variety X_Ξ is smooth except the finite number isolated singularities, $x_{\diamond^c}$s. The structure of X_Ξ near a singular element $x_{\diamond^c}$ can be determined in the following manner. For a given $\diamond^c$, one can construct a tetrahedron Δ^c inside Δ with the core $\diamond^c$ adjacent to four elements Δ^c_j $(1 \leq j \leq 4)$ in $\Xi(3)$ of type Δ_u or Δ_d,

$$
\Delta^c = \diamond^c \cup \bigcup_{j=1}^{4} \Delta^c_j \subseteq \Delta \ ,
$$

such that $\diamond^c \cap \Delta^c_j$ $(1 \leq j \leq 4)$ are four facets of $\diamond^c$, two of which intersect only at one common vertex, (there could have two possible ways of forming such Δ^c with the same core $\diamond^c$). Consider the rational simplicial decomposition Ξ^* of Δ, which is a refinement of Ξ by adding c as a vertex with the barycentric simplicial decomposition $\diamond^c$ for all c. In fact, the octahedron $\diamond^c$ is decomposed into the following eight 4-simplices of Ξ^*:

$$
\begin{array}{lll}
{C_1}^c & := & \langle c, \ \ c+\frac{e^1+e^2-e^3-e^4}{2(r+1)}, \ \ c+\frac{e^1-e^2+e^3-e^4}{2(r+1)}, \ \ c+\frac{e^1-e^2-e^3+e^4}{2(r+1)} \rangle, \\
{C_2}^c & := & \langle c+\frac{e^1+e^2-e^3-e^4}{2(r+1)}, \ \ c, \ \ c+\frac{-e^1+e^2+e^3-e^4}{2(r+1)}, \ \ c+\frac{-e^1+e^2-e^3+e^4}{2(r+1)} \rangle, \\
{C_3}^c & := & \langle c+\frac{e^1-e^2+e^3-e^4}{2(r+1)}, \ \ c+\frac{-e^1+e^2+e^3-e^4}{2(r+1)}, \ \ c, \ \ c+\frac{-e^1-e^2+e^3+e^4}{2(r+1)} \rangle, \\
{C_4}^c & := & \langle c+\frac{e^1-e^2-e^3+e^4}{2(r+1)}, \ \ c+\frac{-e^1+e^2-e^3+e^4}{2(r+1)}, \ \ c+\frac{-e^1-e^2+e^3+e^4}{2(r+1)}, \ \ c \rangle, \\
{C'_1}^c & := & \langle c, \ \ c+\frac{-e^1-e^2+e^3+e^4}{2(r+1)}, \ \ c+\frac{-e^1+e^2-e^3+e^4}{2(r+1)}, \ \ c+\frac{-e^1+e^2+e^3-e^4}{2(r+1)} \rangle, \\
{C'_2}^c & := & \langle c+\frac{-e^1-e^2+e^3+e^4}{2(r+1)}, \ \ c, \ \ c+\frac{e^1-e^2-e^3+e^4}{2(r+1)}, \ \ c+\frac{e^1-e^2+e^3-e^4}{2(r+1)} \rangle, \\
{C'_3}^c & := & \langle c+\frac{-e^1+e^2-e^3+e^4}{2(r+1)}, \ \ c+\frac{e^1-e^2-e^3+e^4}{2(r+1)}, \ \ c, \ \ c+\frac{e^1+e^2-e^3-e^4}{2(r+1)} \rangle, \\
{C'_4}^c & := & \langle c+\frac{-e^1+e^2+e^3-e^4}{2(r+1)}, \ \ c+\frac{e^1-e^2+e^3-e^4}{2(r+1)}, \ \ c+\frac{e^1+e^2-e^3-e^4}{2(r+1)}, c \rangle.
\end{array} \tag{24}
$$

All vertices appeared in the above simplices are elements in $N \cap \Delta$ except c , while $2c \in N$. (see Fig. 5)

One can determine the singularity structure of the variety X_Ξ near $x_{\diamond^c}$ by examining the toric orbits associated to Δ^c. The toric data in $\mathbb{R}^4$ for the lattice N and the cone generated by Δ^c are isomorphic to the toric data of the lattice for the group $A_1(4)$ with the first quadrant cone in Lemma 3.1. Hence as toric varieties, the structure of X_Ξ near the singularity $x_{\diamond^c}$ is the same as that for $A_1(4)$. One can apply the result of Theorem 4.1 to describe the local structure of X_{Ξ^*} over the singular point $x_{\diamond^c}$ of X_Ξ. Hence one concludes that X_{Ξ^*} is a smooth toric variety with the canonical bundle, $\omega_{X_{\Xi^*}} = \mathcal{O}_{X_{\Xi^*}}(\sum_{\diamond^c \in \Xi(4)} E_c)$, where E_c is the toric divisor associated to the vertex c in X_{Ξ^*}, and it satisfies the properties (15)(17). By (7) and the structure of E_c, one obtains

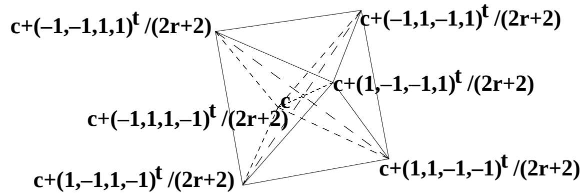

Figure 5: Local figure of the decomposition of the octahedron in the right one of Fig. 4 by adding c.

the desired crepant resolutions $\widehat{S}_{A_r(4)}$ by blowing-down each E_c to $\mathbb{P}^1 \times \mathbb{P}^1$ as in Theorem 3.2. and different crepant resolutions are connected by flop relation. It remains to show $X_{\Xi^*} \simeq \mathrm{Hilb}^G(\mathbb{C}^4)$, and the total number of $\diamond^c$s is equal to $\frac{r(r+1)(r+2)}{6}$. As in the proof of Theorem 3.1, we first construct a regular morphism λ from X_{Ξ^*} to $\mathrm{Hilb}^G(\mathbb{C}^4)$ by examining $I(y)$ for $y \in X_{\Xi^*}$ in terms of toric coordinates. For $\mathfrak{R} \in \Xi^*(3)$, we denote $x_{\mathfrak{R}} := \mathrm{orb}(\mathfrak{R}) \in X_{\Xi^*}$. For the simplicity of notions, we again work on some special 3-simplices as the model cases, whose argument can equally be applied to all elements in $\Xi^*(3)$. We consider the 3-simplices of X_{Ξ^*} contained in the first three polytopes in (23), and they are: Δ_u, Δ_d of (23) and the eight simplices of (24) with $c = \mathbf{v}^m + \frac{e^2+e^4-e^1-e^3}{2(r+1)}$. The affine toric coordinates for X_{Ξ^*} are determined by the integral basis of M in the simplicial cone dual to the one in N generated by the corresponding 3-simplex. By computation, the affine coordinate systems corresponding to these 3-simplices are as follows:

$$\Delta_u : \quad (V_1^{(m_1)}, V_2^{(m_2)}, V_3^{(m_3-1)}, V_4^{(m_4)}), \qquad V_i^{(l)} := \frac{{Z_i}^{r+1-l}}{(Z_1..\breve{Z}_i..Z_4)^l},$$

$$\Delta_d : \quad (V_1'^{(m_1)}, V_2'^{(m_2+1)}, V_3'^{(m_3)}, V_4'^{(m_4)}), \qquad V_i'^{(l)} := \frac{(Z_1..\breve{Z}_i..Z_4)^l}{{Z_i}^{r+1-l}},$$

$$C_i{}^c : \quad (U_{i,1}^{(c)},\ U_{i,2}^{(c)},\ U_{i,3}^{(c)},\ U_{i,4}^{(c)}), \qquad U_{i,i}^{(c)} := \frac{(Z_jZ_jZ_k)^{(r+1)c_i+\frac{1}{2}}}{{Z_i}^{(r+1)(1-c_i)-\frac{1}{2}}},\ U_{i,j}^{(c)} := \frac{(Z_iZ_j)^{(r+1)(1-c_i-c_j)}}{Z_k{Z_s}^{(r+1)(c_i+c_j)}},$$

$$C_i'^c : \quad (U_{1,i}'^{(c)},\ U_{2,i}'^{(c)},\ U_{3,i}'^{(c)},\ U_{4,i}'^{(c)}), \qquad U_{i,i}'^{(c)} := \frac{{Z_i}^{(r+1)(1-c_i)+\frac{1}{2}}}{(Z_jZ_kZ_s)^{(r+1)c_i-\frac{1}{2}}},\ U_{i,j}'^{(c)} := \frac{(Z_kZ_s)^{(r+1)(1-c_k-c_s)}}{(Z_iZ_j)^{(r+1)(c_k+c_s)}}.$$

Here the indices i, j, k, s indicate the four 3 by permuting $1, 2, 3, 4$, and we shall adopt this convention for the rest of this proof if no confusion will arise. Define the following eigen-polynomials of G for $\beta \in \mathbb{C}$ and integers l with $0 \leq l \leq (r+1)$,

$$F_i^{(l)}(\beta) = {Z_i}^l - \beta(Z_jZ_kZ_s)^{(r+1)-l}, G_{i,j}^{(l)}(\beta) = (Z_iZ_j)^l - \beta(Z_kZ_s)^{(r+1)-l}, H_i^{(l)}(\beta) = (Z_jZ_jZ_s)^l - \beta {Z_i}^{(r+1)-l}.$$

Let y be an element of X_{Ξ^*}. For y near x_{Δ_u} with coordinates $(V_1^{(m_1)}, V_2^{(m_2)}, V_3^{(m_3-1)}, V_4^{(m_4)}) = (v_1, v_2, v_3, v_4)$, the ideal $I(y)$ has the generators,

$$\begin{array}{l} F_1^{(r+1-m_1)}(v_1),\ F_2^{(r+1-m_2)}(v_2),\ F_3^{(r+2-m_3)}(v_3),\ F_4^{(r+1-m_4)}(v_4),\ G_{1,2}^{(m_3+m_4)}(v_1v_2), \\ G_{1,3}^{(m_2+m_4+1)}(v_1v_3),\ G_{1,4}^{(m_2+m_3)}(v_1v_4),\ G_{2,3}^{(m_1+m_4+1)}(v_2v_3),\ G_{2,4}^{(m_1+m_3)}(v_2v_4),\ G_{3,4}^{(m_1+m_2+1)}(v_3v_4), \\ H_1^{(m_1+1)}(v_2v_3v_4),\ H_2^{(m_2+1)}(v_1v_3v_4),\ H_3^{(m_3)}(v_1v_2v_4),\ H_4^{(m_4+1)}(v_1v_2v_3),\ Z_1Z_2Z_3Z_4 - v_1v_2v_3v_4. \end{array} \quad (25)$$

For y near x_{Δ_d} with coordinates $(V_1'^{(m_1)}, V_2'^{(m_2+1)}, V_3'^{(m_3)}, V_4'^{(m_4)}) = (v_1', v_2', v_3', v_4')$, $I(y)$ has the generators:

$$\begin{array}{l} F_1^{(r+2-m_1)}(v_2'v_3'v_4'),\ F_2^{(r+1-m_2)}(v_1'v_3'v_4'),\ F_3^{(r+2-m_3)}(v_1'v_2'v_4'),\ F_4^{(r+2-m_4)}(v_1'v_2'v_3'), \\ G_{1,2}^{(m_3+m_4)}(v_3'v_4'),\ G_{1,3}^{(m_2+m_4+1)}(v_2'v_4'),\ G_{1,4}^{(m_2+m_3+1)}(v_2'v_3'),\ G_{2,3}^{(m_1+m_4)}(v_1'v_4'),\ G_{2,4}^{(m_1+m_3)}(v_1'v_3'), \\ G_{3,4}^{(m_1+m_2+1)}(v_1'v_2'),\ H_1^{(m_1)}(v_1'),\ H_2^{(m_2+1)}(v_2'),\ H_3^{(m_3)}(v_3'),\ H_4^{(m_4)}(v_4'),\ Z_1Z_2Z_3Z_4 - v_1'v_2'v_3'v_4'. \end{array} \tag{26}$$

For y near $x_{C_i^c}$ with coordinates $(U_{il}^{(c)} = u_l)_{1\leq l\leq 4}$, $I(y)$ has the generators:

$$\begin{array}{l} F_i^{((r+1)(1-c_i)+\frac{1}{2})}(u_1u_2u_3u_4),\ F_j^{((r+1)(1-c_j)+\frac{1}{2})}(u_iu_j),\ F_k^{((r+1)(1-c_k)+\frac{1}{2})}(u_iu_k), \\ F_s^{((r+1)(1-c_s)+\frac{1}{2})}(u_iu_s),\ G_{i,j}^{(r+1)(c_k+c_s)}(u_j),\ G_{i,k}^{(r+1)(c_j+c_s)}(u_k),\ G_{i,s}^{(r+1)(c_j+c_k)}(u_s), \\ G_{j,k}^{(r+1)(c_i+c_s)+1}({u_i}^2u_ju_k),\ G_{j,s}^{(r+1)(c_i+c_k)+1}({u_i}^2u_ju_s),\ G_{k,s}^{(r+1)(c_i+c_j)+1}({u_i}^2u_ku_s), \\ H_i^{((r+1)c_i+\frac{1}{2})}(u_i),\ H_j^{((r+1)c_j+\frac{1}{2})}(u_iu_ku_s),\ H_k^{((r+1)c_k+\frac{1}{2})}(u_iu_ju_s),\ H_s^{((r+1)c_s+\frac{1}{2})}(u_iu_ju_k), \\ Z_1Z_2Z_3Z_4 - {u_i}^2u_ju_ku_s. \end{array} \tag{27}$$

For y near $x_{C_i'^c}$ with the coordinates $(U_{il}'^{(c)} = u_l')_{1\leq l\leq 4}$, $I(y)$ has the generators:

$$\begin{array}{l} F_i^{((r+1)(1-c_i)+\frac{1}{2})}(u_i')\ , F_j^{((r+1)(1-c_j)+\frac{1}{2})}(u_i'u_k'u_s'),\ F_k^{((r+1)(1-c_k)+\frac{1}{2})}(u_i'u_j'u_s'),\ F_s^{((r+1)(1-c_s)+\frac{1}{2})}(u_i'u_j'u_k'), \\ G_{i,j}^{(r+1)(c_k+c_s)+1}(u_i'^2u_k'u_s'),\ G_{i,k}^{(r+1)(c_j+c_s)+1}(u_i'^2u_j'u_s'),\ G_{i,s}^{(r+1)(c_j+c_k)+1}(u_i'^2u_j'u_k'), \\ G_{j,k}^{(r+1)(c_i+c_s)}(u_s'),\ G_{j,s}^{(r+1)(c_i+c_k)}(u_k'),\ G_{k,s}^{(r+1)(c_i+c_j)}(u_j'), \\ H_i^{((r+1)c_i+\frac{1}{2})}(u_1'u_2'u_3'u_4'),\ H_j^{((r+1)c_j+\frac{1}{2})}(u_i'u_j'),\ H_k^{((r+1)c_k+\frac{1}{2})}(u_i'u_k'),\ H_s^{(c_s+\frac{1}{2})}(u_i'u_s'), \\ Z_1Z_2Z_3Z_4 - u_i'^2u_j'u_k'u_s'. \end{array} \tag{28}$$

The centers of the above affine charts have the monomial ideals , say the one near x_{Δ_u}, $I(x_{\Delta_u})$ is obtained by setting $v_l = 0$ in (25), hence an monomial ideal. There are exactly $(r+1)^3$ monomials not in $I(x_{\Delta_u})$, i.e., $|I(x_{\Delta_u})^\dagger| = (r+1)^3$. For y near x_{Δ_u}, by using (25) and employing the Gröbner basis techniques and the toric data, one obtains the colength of $I(y)$ in $\mathbb{C}[Z]$ satisfying the relation, $\mathrm{colength}(I(y)) \leq \mathrm{colength}(I(x_{\Delta_u})) = (r+1)^3$; this implies $\mathrm{colength}(I(y)) = (r+1)^3$. By which it determines an element $\lambda(y) \in \mathrm{Hilb}^G(\mathbb{C}^4)$. One can also show the colength of $I(y)$ equal to $(r+1)^3$ for y in other affine charts using (26) (27) (28). The same conclusion holds for y in any affine coordinate neighborhood centered at $x_{\mathfrak{R}}$ for $\mathfrak{R} \in \Xi^*(3)$, and one obtains an element $\lambda(y)$ in $\mathrm{Hilb}^G(\mathbb{C}^4)$, by which the morphism $\lambda : X_{\Xi^*} \longrightarrow \mathrm{Hilb}^G(\mathbb{C}^4)$ is defined.

Now we are going to show that λ is an isomorphism. For $n \in \mathbb{Z}$, we denote $\underline{n}$ the unique integer satisfying the relation,

$$n \equiv \underline{n} \pmod{r+1}\ , \quad 0 \leq \underline{n} \leq r\ .$$

We first determine the G-invariant monomial ideals J_0 in $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$. For a such J_0, the set $J_0^\dagger := W_0 \setminus (W_0 \cap J_0)$ forms a basis of a G-regular representation space. Denote l_i the smallest integer with $Z_i^{l_i} \in J_0$; l_{ij} the smallest one with $(Z_iZ_j)^{l_{ij}} \in J_0$ for $i \neq j$, and so on. By $1 \not\in J_0$, and $1 \sim {Z_i}^{r+1} \sim Z_1Z_2Z_3Z_4$, we have ${Z_i}^{r+1}, Z_1Z_2Z_3Z_4 \in J_0$, i.e. $I(o) \subset J_0$, and the following relations hold,

$$1 \leq l_{ijk} \leq l_{ij} \leq l_i \leq r+1\ .$$

By $J_0^\perp \subset I(o)^\perp$, and (9) for the description of the G-eigenspace of $I(o)^\perp$, $(Z_jZ_kZ_s)^{r+1-l_i}$ is the only monomial $u \in I(o)^\dagger$ with $u \sim Z_i^{l_i}$, which implies $(Z_jZ_kZ_s)^{r+1-l_i} \in J_0^\dagger$ and $(Z_jZ_kZ_s)^{r+2-l_i} \in J_0$, hence $l_{jks} = r+2-l_i$. By a similar argument, one has $l_{ks} = r+2-l_{ij}$. Hence we have

$$l_i + l_{jks} = l_{ij} + l_{ks} = r+2\ . \tag{29}$$

We claim that J_0 is the ideal with generators given by

$$J_0 = \langle {Z_i}^{l_i},\ (Z_iZ_j)^{l_{ij}},\ (Z_iZ_jZ_k)^{l_{ijk}},\ Z_1Z_2Z_3Z_4\ \mid i,j,k\rangle\ . \tag{30}$$

(Note that i, j, k are distinct numbers among $1, 2, 3, 4$ as before). Let J_0' the ideal in the right hand side of (30). Then $I(o) \subset J_0' \subset J_0$. Suppose $J_0' \neq J_0$, equivalently $J_0 \cap J_0'^{\dagger} \neq \emptyset$. For the convenience of notations but without loss of generality, we may assume $Z_2^{i_2}Z_3^{i_3}Z_4^{i_4} \in J_0 \cap J_0'^{\dagger}$ for $i_2 \leq i_3 \leq i_4$. Hence $i_2 < l_{234}, i_3 < l_{34}, i_4 < l_4$, which implies p_1 $(:= Z_2^{l_{234}-1}Z_3^{l_{34}-1}Z_4^{l_4-1}) \in J_0 \cap I(0)^{\dagger}$. By (9), the rest of monomials p in $I(o)^{\dagger}$ with $p \sim p_1$ are given by

$$p_2 := Z_1^{\underline{r+2-l_{234}}}Z_3^{l_{34}-l_{234}}Z_4^{l_4-l_{234}}\ ,\quad p_3 := Z_1^{\underline{r+2-l_{34}}}Z_2^{\underline{r+1+l_{234}-l_{34}}}Z_4^{l_4-l_{34}},$$
$$p_3 := Z_1^{\underline{r+2-l_4}}Z_2^{\underline{r+1+l_{234}-l_4}}Z_3^{\underline{r+1+l_{34}-l_4}},$$

among which exactly only one belongs to $J_0^{\dagger}$. We have $p_1 = p_2$ when $l_{234} = 1$. If $l_{234} > 1$, by (29) we have $\underline{r+2-l_{234}} = l_1$. Therefore $p_2 \in J_0$. When $l_{234} = l_{34}$, we have $p_2 = p_3$. When $l_{234} < l_{34}$, $p_3 = (Z_1Z_2)^{l_{1,2}}Z_2^{l_{123}}Z_4^{l_4-l_{34}}$ by (29), hence $p_3 \in J_0$. Similarly, $p_3 = p_4$ when $l_{34} = l_4$. If $l_{34} < l_4$, $u_4 = (Z_1Z_2Z_3)^{l_{123}}Z_2^{l_{234}}Z_3^{l_{34}}$, hence $p_4 \in J_0$. Therefore $p_i \in J_0$ for $1 \leq i \leq 4$, a contradiction to their relations with $J_0^{\dagger}$. We are going to show the following relations hold for $i \neq j$,

$$r+1 \leq l_i + l_j - l_{ij} \leq r+2\ . \tag{31}$$

Consider the element w $(:= Z_i^{l_i}Z_j^{l_{ij}-1}Z_k^{l_{ijk}-1})$ in J_0. Among the following monomials G-equivalent to w,

$$w_1 = Z_i^{\underline{l_i-l_{ijk}+1}}Z_j^{l_{ij}-l_{ijk}}Z_s^{\underline{r+2-l_{ijk}}}\ ,\quad w_2 = Z_i^{\underline{l_i-l_{ij}+1}}Z_k^{\underline{r+1-l_{ij}+l_{ijk}}}Z_s^{\underline{r+2-l_{ij}}}\ ,$$
$$w_3 = Z_j^{r-l_i+l_{ij}}Z_k^{r-l_i+l_{ijk}}Z_s^{r+1-l_i},$$

there exists exactly one in $J_0^{\dagger}$. It is easy to see that $w_1 = Z_i^{\underline{l_i-l_{ijk}+1}}Z_j^{l_{ij}-l_{ijk}}Z_s^{l_s} \in J_0$ unless $l_{ijk} = 1$, in which case $w_1 = w \in J_0$ if $l_i < r+1$, and $w_1 = w_3$ if $l_i = r+1$. We have $w_1 = w_2$ if $l_{ij} = l_{ijk}$. When $l_{ij} > l_{ijk}$, $w_2 = Z_i^{\underline{l_i-l_{ij}+1}}Z_k^{l_{ks}+l_{ijk}-1}Z_s^{l_{ks}} \in J_0$. Therefore w_3 is the element of $J_0^{\dagger}$ G-equivalent to w, which by the expression of the power of Z_j, implies

$$r+1 \leq l_i + l_j - l_{ij}\ .$$

As a consequence of the above inequality, we have $l_j = r+1$ and $l_i + l_j - l_{ij} = r+1$ when $l_{ij} = l_i$, in particular (31) holds. Hence we may assume $l_{ij} < l_i$, in which case $h := Z_i^{l_i-1}Z_j^{l_{ij}}Z_k^{l_{ijk}-1} \in J_0$. Among the following monomials G-equivalent to h,

$$h_1 = Z_i^{l_i-l_{ijk}}Z_j^{l_{ij}-l_{ijk}+1}Z_s^{\underline{r+2-l_{ijk}}}\ ,\quad h_2 = Z_i^{l_i-l_{ij}-1}Z_k^{r-l_i+l_{ijk}}Z_s^{r+1-l_{ij}},$$
$$h_3 = Z_j^{\underline{r+2-l_i+l_{ij}}}Z_k^{r+1-l_j+l_{ijk}}Z_s^{r+2-l_i}\ ,$$

there exists exactly one in $J_0^{\dagger}$. We have $h_1 = h \in J_0$ if $l_{ijk} = 1$. When $l_{ijk} > 1$, $h_1 = Z_i^{l_i-l_{ijk}}Z_j^{l_{ij}-l_{ijk}+1}Z_s^{l_s}$, and $h_1 \in J_0$. One has $h_3 = Z_j^{l_{ij}}Z_k^{l_{ijk}-1}(Z_jZ_kZ_s)^{l_{jks}} \in J_0$ unless $l_i = l_{ij}+1$, in which case, $h_3 = h_2$. Therefore we have $h_2 \in J_0^{\dagger}$, which implies $l_i - l_{ij} - 1 \leq l_{iks} - 1$, hence $l_i + l_j - l_{ij} \leq r+2$ by (29). Therefore we obtain the relation (31). With $(i,j) = (1,2), (3,4)$ in (31) (29), we have $3r+4 \leq \sum_{j=1}^4 l_j \leq 3r+6$. Using (29), one obtains the all possible cases of $l_i + l_j - l_{ij}$ for a given value of $\sum_{j=1}^4 l_j$; consequently, all the values of l_Is are determined by l_is.

By comparing the polynomials at the origin in (25) (26) (27) (28), $J_0 = I(x_{\mathfrak{R}})$ for $\mathfrak{R} \in \Xi^*(3)$ by the following relations:

$$\begin{array}{ll} \Delta_u: & \sum_{j=1}^4 l_j = 3r+4, \quad l_{ij} = l_i + l_j - r - 1; \\ \Delta_d: & \sum_{j=1}^4 l_j = 3r+6, \quad l_{ij} = l_i + l_j - r - 2; \\ C_i^c: & \sum_{j=1}^4 l_j = 3r+5, \quad l_{ij} = l_i + l_j - r - 2 \ , \ l_{ks} = l_k + l_s - r - 1; \\ C_i'^c: & \sum_{j=1}^4 l_j = 3r+5, \quad l_{ij} = l_i + l_j - r - 1 \ , \ l_{ks} = l_k + l_s - r - 2 \ , \end{array} \tag{32}$$

where the indices in toric data are connected to the l_is by the following relations,

$$\begin{array}{ll} \Delta_u: & l_3 = r+2-m_3, \ l_j = r+1-m_j, \ (j \neq 3) \ , \\ \Delta_d: & l_2 = r+1-m_2, \ l_j = r+2-m_j, \ (j \neq 2) \ , \\ C_i^c, C_i'^c: & l_j = (r+1)(1-c_j) + \frac{1}{2} \ , \quad c = \frac{1}{2r+2} \sum_{j=1}^4 (2r+3-2l_j) e^j \ . \end{array}$$

With $l'_i := r+1-l_i$, l'_is are 4 positive integers satisfying the equation $\sum_{i=1}^4 l'_i = L'$ with $L' = r, r-1, r-2$. The number of solutions of l'_is is equal to $\begin{pmatrix} L'+3 \\ 3 \end{pmatrix}$. Hence one obtains the following numbers of $\mathfrak{R} \in \Xi^*(3)$ for the toric data in (23) (24) using the relation with l_is:

$$\#\{\Delta_u\} = \frac{(r+1)(r+2)(r+3)}{6}, \quad \#\{\Delta_d\} = \frac{(r-1)r(r+1)}{6}, \quad \#\{c\} = \frac{r(r+1)(r+2)}{6}. \tag{33}$$

Let J be a G-invariant ideal representing an element in $\mathrm{Hilb}^G(\mathbb{C}^4)$. With the Gröbner basis argument as in Theorem 3.1, there is a monomial ideal J_0 in $\mathrm{Hilb}^G(\mathbb{C}^4)$ such that $J_0^\dagger$ gives rise to a basis of $\mathbb{C}[Z]/J$ with the relation (22). As $J_0 = I(x_{\mathfrak{R}})$ for some $\mathfrak{R} \in \Xi^*(3)$, which is determined by the integers l_i, l_{ij}, l_{ijk} with the relations in (29) (32), this implies that for some $\gamma_i, \gamma_{ij}, \gamma_{jks}, \gamma_{1234} \in \mathbb{C}$, the polynomials $F_i^{(l_i)}(\gamma_i), G_{ij}^{(l_{ij})}(\gamma_{ij}), H_i^{(l_{jks})}(\gamma_{jks})$ and $Z_1Z_2Z_3Z_4 - \gamma_{1234}$ are elements of J. From the expressions of $F_i^{(l)}(\beta), G_{i,j}^{(l)}(\beta), H_i^{(l)}(\beta)$, and using $\dim(\mathbb{C}[Z]/J) = (r+1)^3$, one can conclude

$$J = \langle F_i^{(l_i)}(\gamma_i), G_{ij}^{(l_{ij})}(\gamma_{ij}), H_i^{(l_{jks})}(\gamma_{jks}), Z_1Z_2Z_3Z_4 - \gamma_{1234} \rangle_{i,j,k,s}$$

We are going to determine the relations among the γ_Is using the relations (29)(32) and according to the type of l_is. By

$$(\gamma_{1234} - \gamma_{123}\gamma_4){Z_4}^{l_4-1} = Z_1Z_2Z_3F_4^{(l_4)}(\gamma_4) - \gamma_4 H_4^{(l_{123})}(\gamma_{123}) - {Z_4}^{l_4-1}(Z_1Z_2Z_3Z_4 - \gamma_{1234}) \in J,$$

and ${Z_4}^{l_4-1} \notin J$, we have $\gamma_{1234} = \gamma_{123}\gamma_4$.

For J with J_0 of type Δ_u, by (32) we have

$$\begin{array}{l} (\gamma_{123} - \gamma_{12}\gamma_3){Z_3}^{l_{34}-1}{Z_4}^{l_4-1} = (Z_1Z_2)^{l_{123}}F_3^{(l_3)}(\gamma_3) + \gamma_3 {Z_4}^{l_{124}-1}G_{12}^{(l_{12})}(\gamma_{12}) - {Z_3}^{l_{34}-1}H_4^{(l_{123})}(\gamma_{123}), \\ (\gamma_{13} - \gamma_1\gamma_3){Z_3}^{l_{234}-1}(Z_2Z_4)^{l_{24}-1} = \gamma_3(Z_2Z_4)^{l_{124}-1}F_1^{(l_1)}(\gamma_1) + {Z_1}^{l_{13}}F_3^{(l_3)}(\gamma_3) - {Z_3}^{l_3-l_{13}}G_{13}^{(l_{13})}(\gamma_{13}), \end{array}$$

which are elements in J. By ${Z_3}^{l_{34}-1}{Z_4}^{l_4-1}, {Z_3}^{l_{234}-1}(Z_2Z_4)^{l_{24}-1} \in J_0^\dagger$, we have $\gamma_{123} = \gamma_1\gamma_{23}, \gamma_{23} = \gamma_2\gamma_3$. By permuting the indices, one obtains $\gamma_I = \prod_{i \in I} \gamma_i$ for a subset I of $\{1,2,3,4\}$. By (25)(33), we have $J = I(y)$ for y near x_{Δ_u} with the coordinates $v_i = \gamma_i$.

When J_0 is of type Δ_u, by (32), the following elements are in J,

$$\begin{array}{l} (\gamma_{12}\gamma_{134} - \gamma_1){Z_2}^{l_{234}-1}(Z_3Z_4)^{l_{34}-1} = (Z_3Z_4)^{l_{134}}F_1^{(l_1)}(\gamma_1) - \gamma_{134}{Z_2}^{l_{234}-1}G_{12}^{(l_{12})}(\gamma_{12}) - {Z_1}^{l_{12}}H_2^{(l_{134})}(\gamma_{134}), \\ (\gamma_{12} - \gamma_{123}\gamma_{124}){Z_3}^{l_3-1}{Z_4}^{l_{34}-1} = -{Z_3}^{l_{123}}G_{12}^{(l_{12})}(\gamma_{12}) + \gamma_{123}{Z_4}^{l_{34}-1}H_3^{(l_{124})}(\gamma_{124}) + (Z_1Z_2)^{l_{124}}H_4^{(l_{123})}(\gamma_{123}) \ . \end{array}$$

Therefore $\gamma_1 = \gamma_{12}\gamma_{134}$ and $\gamma_{12} = \gamma_{123}\gamma_{124}$. Set $v'_i = \gamma_{1..\check{i}..4}$. With the same argument, one obtains $\gamma_I = \prod_{j \in I'} v'_j$ for $I \neq 1234$, where I' is the complement set of I in $\{1,2,3,4\}$. Therefore by (26) (33), $J = I(y)$ for y near x_{Δ_d} having v'_i's as coordinates.

When J_0 is of type C^c_i or C'^c_i, without loss of generality, we may assume $i = 1$. In the case C^c_1, the following elements are in J by (32),

$$
\begin{aligned}
&(\gamma_{123} - \gamma_{13}\gamma_2)(Z_1Z_3)^{l_{134}-1}{Z_4}^{l_4-1} = \gamma_{13}{Z_4}^{l_{24}-1}F_2^{(l_2)}(\gamma_2) + {Z_2}^{l_{13}-l_{134}+1}G_{13}^{(l_{13})}(\gamma_{13}) - (Z_1Z_3)^{l_{134}-1}H_4^{(l_{123})}(\gamma_{123}),\\
&(\gamma_2 - \gamma_{12}\gamma_{234})Z_1^{l_{134}-1}(Z_3Z_4)^{l_{34}-1} = -(Z_3Z_4)^{l_{234}}F_2^{(l_2)}(\gamma_2) + \gamma_{234}{Z_1}^{l_{134}-1}G_{12}^{(l_{12})}(\gamma_{12}) + Z_2^{l_{12}}H_1^{(l_{234})}(\gamma_{234}),\\
&(\gamma_1 - \gamma_{12}\gamma_{134})Z_2^{l_{234}-1}(Z_3Z_4)^{l_{34}-1} = -(Z_3Z_4)^{l_1-l_{1,2}}F_1^{(l_1)}(\gamma_1) + \gamma_{134}{Z_2}^{l_{234}-1}G_{12}^{(l_{12})}(\gamma_{1,2}) + {Z_1}^{l_{12}}H_2^{(l_{134})}(\gamma_{134}),\\
&(\gamma_{23} - \gamma_2\gamma_3)(Z_2)^{l_{124}-1}(Z_1Z_4)^{l_{14}-1} = \gamma_2(Z_1Z_4)^{l_3-l_{23}}F_3^{(l_3)}(\gamma_3) + {Z_3}^{l_{23}}F_2^{(l_2)}(\gamma_2) - {Z_2}^{l_{124}-1}G_{23}^{(l_{23})}(\gamma_{23}) \ .
\end{aligned}
$$

Hence

$$
\gamma_{123} = \gamma_2\gamma_{13} \ , \gamma_2 = \gamma_{234}\gamma_{12} \ , \gamma_1 = \gamma_{12}\gamma_{134} \ , \gamma_{23} = \gamma_2\gamma_3,
$$

which are the same relations as u_Is in (27) for $i = 1$ under the identification: $u_1 = \gamma_{234}$, and $u_j = \gamma_{1j}$ for $j \neq 1$. By permuting the indices, one can show that all the rest relations in (27) are satisfied in terms of the γ_Is. Hence by (33), $J = I(y)$ for y near x_{C_1} with u_is as the coordinates of y.

For J_0 is of type C'^c_1, the following elements are in J by (32),

$$
\begin{aligned}
&(\gamma_{234} - \gamma_{34}\gamma_2){Z_1}^{l_1-1}{Z_2}^{l_{12}-1} = (Z_3Z_4)^{l_{234}}F_2^{(l_2)}(\gamma_2) + \gamma_2{Z_1}^{l_1-l_{12}}G_{34}^{(l_{34})}(\gamma_{34}) - Z_2^{l_{12}-1}H_1^{(l_{234})}(\gamma_{234}),\\
&(\gamma_2 - \gamma_{23}\gamma_{124})(Z_1Z_4)^{l_{134}-1}{Z_3}^{l_3-1} = -{Z_3}^{l_{23}}F_2^{(l_2)}(\gamma_2) + {Z_2}^{l_{124}}G_{23}^{l_{2,3}}(\gamma_{23}) + \gamma_{23}(Z_1Z_4)^{l_{134}-1}H_3^{(l_{124})}(\gamma_{124}),\\
&(\gamma_{124} - \gamma_1\gamma_{24}){Z_1}^{l_{13}-1}{Z_3}^{l_3-1} = (Z_2Z_4)^{l_{124}}F_1^{(l_1)}(\gamma_1) + \gamma_1{Z_3}^{l_3-l_{1,3}}G_{24}^{(l_{24})}(\gamma_{24}) - {Z_1}^{l_{1,3}-1}H_3^{(l_{124}}(\gamma_{124}),\\
&(\gamma_{12} - \gamma_1\gamma_2){Z_2}^{l_{234}-1}(Z_3Z_4)^{l_{34}-1} = \gamma_2(Z_3Z_4)^{l_1-l_{12}}F_1^{(l_1)}(\gamma_1) + {Z_1}^{l_{12}}F_2^{(l_2)}(\gamma_2) - {Z_2}^{l_{234}-1}G_{12}^{l_{12}}(\gamma_{12}).
\end{aligned}
$$

Hence

$$
\gamma_{234} = \gamma_{34}\gamma_2, \ \gamma_2 = \gamma_{23}\gamma_{124}, \ \gamma_{124} = \gamma_1\gamma_{24}, \ \gamma_{12} = \gamma_1\gamma_2 \ ,
$$

which are the same relations of u'_Is in (28) for $i = 1$ under the identification: $u'_1 = \gamma_1, u'_2 = \gamma_{34}, u'_3 = \gamma_{24}, u'_4 = \gamma_{23}$. By the similar argument, all the relations of (28) hold; therefore $J = I(y)$ for y near $x_{C'_1}$ having the coordinates u'_is.

By the results we have obtained, one concludes that $\mathrm{Hilb}^G(\mathbb{C}^4)$ is a smooth toric variety, hence of the form $X_{\Xi^{**}}$ where Ξ^{**} is a simplicial decomposition of Δ which is refinement of Ξ^* corresponding to the morphism λ. Indeed, the above analysis of local structure of $\mathrm{Hilb}^G(\mathbb{C}^4)$ has shown $\Xi^* = \Xi^{**}$, therefore λ is an isomorphism between X_{Ξ^*} and $\mathrm{Hilb}^G(\mathbb{C}^4)$. The number of exceptional divisors appearing in the canonical bundle of X_{Ξ^*} is equal to $\frac{r(r+1)(r+2)}{6}$ by (33). $\square$

5 G-Hilbert Scheme over $\mathbb{C}^3/\mathfrak{A}_4$

It is known that the alternating group $\mathfrak{A}_{n+1}$ is a simple group except $n = 2, 3$, in which cases, $\mathfrak{A}_3 \simeq \mathbb{Z}_3$ and $\mathfrak{A}_4$ is isomorphic to the ternary trihedral group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \lhd \mathbb{Z}_3$. The G-Hilbert scheme for $\mathfrak{A}_3$ is the minimal resolution of $\mathbb{C}^2/\mathfrak{A}_3$. In this section we are going to give a constructive proof of the smooth and explicit crepant structure of $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$.

Theorem **5.1** $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ *is a smooth variety with trivial canonical bundle.*

We shall devote the rest of this section to the proof of the above theorem, and always denote $G = \mathfrak{A}_4$. Introduce the following coordinates (z_1, z_2, z_3) of V in $(10)_{n=3}$,

$$
z_1 = -\widetilde{z}_1 + \widetilde{z}_2 + \widetilde{z}_3 - \widetilde{z}_4, \quad z_2 = \widetilde{z}_1 - \widetilde{z}_2 + \widetilde{z}_3 - \widetilde{z}_4, \quad z_3 = \widetilde{z}_1 + \widetilde{z}_2 - \widetilde{z}_3 - \widetilde{z}_4 \ ,
$$

where $\sum_{j=1}^{4} \overline{z}_j = 0$. The irreducible representation of G on $\mathbb{C}^3 (= V)$, denoted by $\mathbf{3}$, has the following matrix forms for generators of G,

$$(12)(34) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (13)(24) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ (123) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

There are 4 distinct irreducible G-modules, $\mathrm{Irr}(G) = \{\mathbf{1}, \mathbf{1}_\omega, \mathbf{1}_{\overline{\omega}}, \mathbf{3}\}$, where $\omega := e^{\frac{2\pi\sqrt{-1}}{3}}$, and $\mathbf{1}_*$ is the G-character determined only by the (123)-value $*$. Using the coordinates $(z_i)_{i=1}^3$ of $\mathbb{C}^3$, the generators of G-invariant polynomials in $\mathbb{C}[Z]$ are:

$$\begin{array}{llll} Y_1 & = Z_1^2 + Z_2^2 + Z_3^2, & Y_2 & = Z_1 Z_2 Z_3, \\ Y_3 & = Z_1^2 Z_2^2 + Z_2^2 Z_3^2 + Z_3^2 Z_1^2, & X & = (Z_1^2 - Z_2^2)(Z_2^2 - Z_3^2)(Z_3^2 - Z_1^2) . \end{array}$$

Note that the above variables are related to s_2, s_3, s_4, d in $(11)_{n=3}$ by the relations, $Y_1 = -8s_2$, $Y_2 = -8s_3$, $Y_3 = 16s_2^2 - 64s_4$, $X = 64d$. The G-invariant polynomial relation (11) with F_3 in (12) becomes

$$X^2 = -4Y_1^3 Y_2^2 - 27Y_2^4 + 18Y_1 Y_2^2 Y_3 + Y_1^2 Y_3^2 - 4Y_3^3 . \tag{34}$$

Let $\mathbb{C}[Z]_j$ be the space of homogeneous polynomials of degree j, and denote $I(o)_j^\perp = I(o)^\perp \cap \mathbb{C}[Z]_j$. Then $I(o)_j^\perp$ is a G-submodule of $I(o)^\perp$. In fact, the only non-zero $I(o)_j^\perp$s are among the range $0 \leq j \leq 5$, whose G-irreducible factors are as follows, (an equivalent form see, e.g., Table 2.2 in [6]),

$$\begin{array}{llll} I(o)_0^\perp = m_0 & \simeq & \mathbf{1}, & m_0 = \mathbb{C}, \\ I(o)_1^\perp = m_1 & \simeq & \mathbf{3}, & m_1 = \{Z_1, Z_2, Z_3\}, \\ I(o)_2^\perp = m_2 + m_3 + m_4 & \simeq & \mathbf{1}_{\overline{\omega}} + \mathbf{1}_\omega + \mathbf{3}, & m_2 = \{f\}, \ m_3 = \{\overline{f}\}, \ m_4 = \{Z_2Z_3, Z_3Z_1, Z_1Z_2\}, \\ I(o)_3^\perp = m_5 + m_6 & \simeq & \mathbf{3} + \mathbf{3}, & m_5 = f\{Z_1, \omega^2 Z_2, \omega Z_3\}, m_6 = \overline{f}\{Z_1, \omega Z_2, \omega^2 Z_3\}, \\ I(o)_4^\perp = m_7 + m_8 + m_9 & \simeq & \mathbf{1}_{\overline{\omega}} + \mathbf{1}_\omega + \mathbf{3}, & m_7 = \{\overline{f}^2\}, m_8 = \{f^2\}, m_9 = f\{\omega Z_1 Z_2, Z_2 Z_3, \omega^2 Z_3 Z_1\}, \\ I(o)_5^\perp = m_{10} & \simeq & \mathbf{3}, & m_{10} = \overline{f}^2\{Z_1, \omega^2 Z_2, \omega Z_3\} , \end{array} \tag{35}$$

where $f := \sum_{j=1}^{3} \omega^{j-1} Z_j^2$, $\overline{f} := \sum_{j=1}^{3} \omega^{2j-2} Z_j^2$. We have the G-irreducible decomposition, $I(o)^\perp = \sum_{k=0}^{10} m_k$. Note that $f\overline{f}, f^3, \overline{f}^3$ are G-invariant polynomials with the following relations,

$$f\overline{f} = Y_1^2 - 3Y_3, \quad f^3 - \overline{f}^3 = 3(\omega^2 - \omega)X, \quad f^3 + \overline{f}^3 = 27Y_2^2 - 9Y_1 Y_3 + 2Y_1^3 . \tag{36}$$

Lemma **5.1** *Among m_ks $(1 \leq k \leq 10)$, the following tree diagram holds:*

$$\begin{tikzcd} & m_2 \arrow[r, dash] & m_5 \arrow[r, dash] \arrow[dr, dash] & m_8 \arrow[dr, dash] & \\ m_1 \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & m_4 \arrow[ur, dash] \arrow[dr, dash] & & m_9 \arrow[r, dash] & m_{10} \\ & m_3 \arrow[r, dash] & m_6 \arrow[r, dash] \arrow[ur, dash] & m_7 \arrow[ur, dash] & \end{tikzcd}$$

where the m_j of the right end of an edge is contained in the ideal generated by the m_i of the left end of the edge and $I(o)$.

Proof. By the expression of m_k, all the relations in the above diagram are trivial except the following ones:

$$m_9 \subset m_6 + I(o) \ , \quad m_{10} \subset m_8 + I(o) \ , \quad m_{10} \subset m_9 + I(o) \ . \tag{37}$$

Define the irreducible G-submodules of $\mathbb{C}[Z]$, isomorphic to $\mathbf{3}$: $\overline{m}_9 = \overline{f}\{\omega^2 Z_1 Z_2, Z_2 Z_3, \omega Z_3 Z_1\}$, $\overline{m}_{10} := f^2\{Z_1, \omega Z_2, \omega^2 Z_3\}$. Then we have the equalities of ideals in $\mathbb{C}[Z]$, $\langle m_9, I(o)\rangle = \langle \overline{m}_9, I(o)\rangle$, $\langle m_{10}, I(o)\rangle = \langle \overline{m}_{10}, I(o)\rangle$, which imply the relations in (37). $\Box$

We shall call an ideal J_0 in $\mathrm{Hilb}^G(\mathbb{C}^3)$ to be central if J_0 is generated by $I(o)$ and a finite number of m_ks. (The central ideal J_0 here will play a similar role of monomial ideals in previous sections for the case of abelian group.) By Lemma 5.1, there are exactly four central ideal J_0 with the following G-irreducible decomposition of $\mathbb{C}[Z]/J_0$ presented in (38).

$$\begin{array}{ll} J_0, & \mathbb{C}[Z]/J_0 \\ x_0 := \langle f\rangle + I(o), & m_0+m_1+m_3+m_4+m_6+m_7; \\ x_0' := \langle \overline{f}\rangle + I(o), & m_0+m_1+m_2+m_4+m_5+m_8; \\ x_\infty := \langle Z_1 f, \omega^2 Z_2 f, \omega Z_3 f, \overline{f}^2\rangle + I(o), & m_0+m_1+m_2+m_3+m_4+m_6; \\ x_\infty' := \langle Z_1 \overline{f}, \omega Z_2 \overline{f}, \omega^2 Z_3 \overline{f}, f^2\rangle + I(o)\ , & m_0+m_1+m_2+m_3+m_4+m_5. \end{array} \tag{38}$$

Note that the J_0's presented in (38) are characterized as the ideals in $\mathrm{Hilb}^G(\mathbb{C}^3)$ with monomial polynomial generators in $\mathbb{C}[Z]$. All the above four elements lie over $o \in S_G$ under the morphism σ_{Hilb} of (4). By the analysis in §2.5 of [6], $\sigma_{\mathrm{Hilb}}^{-1}(o)$ consists of a tree of three smooth rational curves, $L + l + L'$. Here are the locations of J_0s in $\sigma_{\mathrm{Hilb}}^{-1}(o)$: $x_0 \in (L\setminus l)\cup L'$, $x_\infty = L\cap l$, $x'_\infty = L'\cap l$, $x'_0 \in (L'\setminus l)\cup L$, (see Fig. 6). We are going to show that every J in $\mathrm{Hilb}^G(\mathbb{C}^3)$ can be deformed to

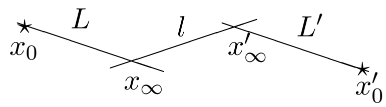

Figure 6: Tree configuration of $\sigma_{\mathrm{Hilb}}^{-1}(o)$ for $G = \mathfrak{A}_4$

one J_0 in (38). For $J \in \mathrm{Hilb}^G(\mathbb{C}^3)$, denote $h(J)$ be the homogenous ideal in $\mathbb{C}[Z]$ generated by the highest total degree part of elements in J. As the top degree of a polynomial in $\mathbb{C}[Z]$ is preserved under the G-action, $h(J)$ is G-invariant. By applying the Gröbner basis technique with a monomial order of total degree in $\mathbb{C}[Z]$, one obtains the same ideal, $\mathrm{lt}(J) = \mathrm{lt}(h(J))$, hence a set of monomial elements in $\mathbb{C}[Z]$ which represent the basis for both $\mathbb{C}[Z]/J$ and $\mathbb{C}[Z]/h(J)$. Therefore $h(J)$ is a homogeneous ideal in $\mathrm{Hilb}^G(\mathbb{C}^3)$. Note that $\sigma_{\mathrm{Hilb}}(h(J)) = o$. By (2.4) in [6], $h(J) \in \{x_0, x_0'\}\cup l$. Hence $h(J)$ and J can be deformed to an element in (38). Now we are going to determine the local structure near these four central elements in $\mathrm{Hilb}^G(\mathbb{C}^3)$.

For J near the element x_∞ in $\mathrm{Hilb}^G(\mathbb{C}^3)$, we have

$$J = \langle \overline{f}^2 - v_0 f,\ m_5 - v_1 m_6 - v_2 m_4 - v_3 m_1,\ Y_1-\eta_1, Y_2-\eta_2, Y_3-\eta_3, X-\xi\rangle\ ,$$

where $(\xi,\eta_1,\eta_2,\eta_3)$ satisfies (34), and $m_5 - v_1 m_6 - v_2 m_4 - v_3 m_1$ is the G-module $\sum_{j=1}^3 \mathbb{C}p_j$ with $p_1 := fZ_1 - v_1\overline{f}Z_1 - v_2 Z_2Z_3 - v_3 Z_1$, $p_2 := f\omega^2 Z_2 - v_1\overline{f}\omega Z_2 - v_2 Z_3 Z_1 - v_3 Z_2$, $p_3 := f\omega Z_3 - v_1\overline{f}\omega^2 Z_3 - v_2 Z_1 Z_2 - v_3 Z_3$. By

$$\begin{array}{rcl} f^2 - (v_1\eta_1 + v_3)\overline{f} &=& Z_1p_1 + \omega^2 Z_2 p_2 + \omega Z_3 p_3 + v_1(Y_1-\eta_1)\overline{f} \in J \\ (\eta_1 - v_3 - v_0 v_1) f &=& Z_1 p_1 + \omega Z_2 p_2 + \omega^2 Z_3 p_3 + v_1(\overline{f}^2 - v_0 f) - (Y_1 - \eta_1) f \in J\ , \end{array} \tag{39}$$

and the first relation of (36), we have $(3\eta_3 - \eta_1^2 - v_0(v_3 + v_1\eta_1))f \in J$. As $f \notin J$, we have

$$\eta_1 - v_3 - v_0 v_1 = 0\ ,\ 3\eta_3 - \eta_1^2 = v_0(v_3 + v_1\eta_1)\ .$$

By the relations: $3Y_2 f - v_2(Z_2^2Z_3^2 + \omega Z_3^2 Z_1^2 + \omega^2 Z_1^2 Z_2^2) = Z_2Z_3p_1 + \omega Z_3 Z_1 p_2 + \omega^2 Z_1 Z_2 p_3 \in J$, and $\overline{f}^2 - Y_1 f = 3(Z_2^2Z_3^2 + \omega Z_3^2Z_1^2 + \omega^2 Z_1^2 Z_2^2)$, we have

$$9\eta_2 + v_2\eta_1 - v_0 v_2 = 0\ .$$

By (36) (39), we have

$$3(\omega^2-\omega)\xi \equiv f^3-\overline{f}^3 \equiv (v_1\eta_1+v_3-v_0)f\overline{f} \equiv (v_1\eta_1+v_3-v_0)(\eta_1^2-3\eta_3) \pmod{J} ,$$

hence

$$3(\omega^2-\omega)\xi = (v_1\eta_1+v_3-v_0)(\eta_1^2-3\eta_3) .$$

By the relation

$$\begin{array}{l} v_2p_3-(\omega-\omega^2)((\omega^2+\omega v_1)Z_1p_2-(\omega+\omega^2v_1)Z_2p_1) \equiv \\ v_1(9\eta_2+v_0v_1v_2+v_2v_3-v_0v_2)Z_3+(3v_1\eta_1+3v_3-3v_1v_3-v_2^2)Z_1Z_2 \qquad \pmod{J} , \end{array}$$

and Z_1Z_2, Z_3 representing two basis elements of $\mathbb{C}[Z]/J$, one obtains

$$3v_1\eta_1+3v_3-3v_1v_3-v_2^2=0 \ , \ v_1(9\eta_2+v_0v_1v_2+v_2v_3-v_0v_2)=0 .$$

With all the above relations among v_js, η_ks and ξ in the above, one can conclude that (v_0, v_1, v_2) forms a coordinate system of $\mathrm{Hilb}^G(\mathbb{C}^3)$ centered at x_∞, and the other parameters in the expression of the ideal J are expressed by the following relations,

$$\begin{array}{lll} v_3 & = \frac{1}{3}v_2^2-v_0v_1^2, & \eta_1=\frac{1}{3}v_2^2+v_0v_1-v_0v_1^2, \\ \eta_2 & = \frac{1}{27}v_2(3v_0-3v_0v_1-v_2^2+3v_0v_1^2), & \eta_3=\frac{1}{27}(3v_0v_1^2-v_2^2)(3v_0v_1^2-3v_0v_1-v_2^2+3v_0), \\ \xi & = \frac{\omega-\omega^2}{81}v_0(v_1+1)(3v_0v_1^2+3v_0-3v_0v_1-v_2^2)(3v_0v_1^3-v_2^2-v_1v_2^2). & \end{array}$$

Note that the above $\xi, \eta_1, \eta_2, \eta_3$ satisfy the relation (34). Furthermore, v_js are G-invariant rational functions in Z_is with the following expressions:

$$\begin{array}{llll} v_0 = & \frac{3(\omega-\omega^2)\xi-9\eta_1\eta_3+27\eta_2^2+2\eta_1^3}{2(\eta_1^2-3\eta_3)}, & v_1 = & \frac{(\omega-\omega^2)\xi+\eta_1\eta_3-9\eta_2^2}{(\omega-\omega^2)\xi-\eta_1\eta_3+9\eta_2^2}, \\ v_2 = & \frac{6\eta_2(\eta_1^2-3\eta_3)}{(\omega-\omega^2)\xi-\eta_1\eta_3+9\eta_2^2}, & v_3 = & \frac{-2\eta_3(\eta_1^2-3\eta_3)}{(\omega-\omega^2)\xi-\eta_1\eta_3+9\eta_2^2}. \end{array}$$

This implies $dZ_1\wedge dZ_2\wedge dZ_3 = \frac{\omega-\omega^2}{36}dv_0\wedge dv_1\wedge dv_2$.

For J near the element x'_∞ in $\mathrm{Hilb}^G(\mathbb{C}^3)$, we have

$$J=\langle f^2-v'_0\overline{f}, \ m_6-v'_1m_5-v'_2m_4-v'_3m_1, \ Y_1-\eta_1, Y_2-\eta_2, Y_3-\eta_3, X-\xi\rangle .$$

By a similar argument as the case x_∞, (v'_0, v'_1, v'_2) forms a coordinate system of $\mathrm{Hilb}^G(\mathbb{C}^3)$ centered at x'_∞ with the relations,

$$\begin{array}{lll} v'_3 & = \frac{1}{3}v_2'^2-v'_0v_1'^2, & \eta_1=\frac{1}{3}v_2'^2+v'_0v'_1-v'_0v_1'^2, \\ \eta_2 & = \frac{1}{27}v_2(3v'_0-3v'_0v'_1-v_2'^2+3v'_0v_1'^2), & \eta_3=\frac{1}{27}(3v'_0v_1'^2-v_2'^2)(3v'_0v_1'^2-3v'_0v'_1-v_2'^2+3v'_0) \\ \xi & = \frac{\omega^2-\omega}{81}v'_0(v'_1+1)(3v'_0v_1'^2+3v'_0-3v'_0v'_1-v_2'^2)(3v'_0v_1'^3-v_2'^2-v'_1v_2'^2) . & \end{array}$$

We have

$$\begin{array}{llll} v'_0 = & \frac{3(\omega^2-\omega)\xi-9\eta_1\eta_3+27\eta_2^2+2\eta_1^3}{2(\eta_1^2-3\eta_3)}, & v'_1 = & \frac{(\omega^2-\omega)\xi+\eta_1\eta_3-9\eta_2^2}{(\omega^2-\omega)\xi-\eta_1\eta_3+9\eta_2^2}, \\ v'_2 = & \frac{6\eta_2(\eta_1^2-3\eta_3)}{(\omega^2-\omega)\xi-\eta_1\eta_3+9\eta_2^2}, & v'_3 = & \frac{-2\eta_3(\eta_1^2-3\eta_3)}{(\omega^2-\omega)\xi-\eta_1\eta_3+9\eta_2^2}, \end{array}$$

and $dZ_1\wedge dZ_2\wedge dZ_3 = \frac{\omega^2-\omega}{36}dv'_0\wedge dv'_1\wedge dv'_2$.

For J near x_0 in $\mathrm{Hilb}^G(\mathbb{C}^3)$, we have

$$J=\langle f-u_0\overline{f}^2, \ \overline{m}_9-u_1m_6-u_2m_4-u_3m_1, \ Y_1-\eta_1, Y_2-\eta_2, Y_3-\eta_3, X-\xi\rangle ,$$

where $(\xi, \eta_1, \eta_2, \eta_3)$ is as before, and $\overline{m}_9 - u_1 m_6 - u_2 m_4 - u_3 m_1$ is the G-module $\sum_{j=1}^3 \mathbb{C} q_j$ with $q_1 := \overline{f} Z_2 Z_3 - u_1 \overline{f} Z_1 - u_2 Z_2 Z_3 - u_3 Z_1$, $q_2 := \overline{f} \omega Z_3 Z_1 - u_1 \overline{f} \omega Z_2 - u_2 Z_3 Z_1 - u_3 Z_2$, $q_3 := \overline{f} \omega^2 Z_1 Z_2 - u_1 \overline{f} \omega^2 Z_3 - u_2 Z_1 Z_2 - u_3 Z_3$. By the relation, $-(u_1 + u_0 u_3)\overline{f}^2 \equiv -u_1 \overline{f}^2 - u_3 f = Z_1 q_1 + \omega Z_2 q_2 + \omega^2 Z_3 q_3 \pmod{J}$, we have

$$u_1 = -u_0 u_3 \ .$$

By $(3\eta_2 - u_1\eta_1 - u_3)\overline{f} = Z_1 q_1 + \omega^2 Z_2 q_2 + \omega Z_3 q_3 \in J$, we have

$$3\eta_2 = u_1 \eta_1 + u_3 = u_3(1 - u_0 \eta_1) \ .$$

By the relations, $f^2 \equiv u_0 \overline{f}^2 f \equiv u_0(\eta_1^2 - 3\eta_3)\overline{f} \pmod{J}$ and

$$\begin{array}{l} Z_2 Z_3 q_1 + \omega^2 Z_3 Z_1 q_2 + \omega Z_1 Z_2 q_3 \equiv (\eta_3 - 3u_1\eta_2)\overline{f} - u_2(Z_2^2 Z_3^2 + \omega^2 Z_3^2 Z_1^2 + \omega Z_1^2 Z_2^2) \pmod{J}, \\ Z_2^2 Z_3^2 + \omega^2 Z_3^2 Z_1^2 + \omega Z_1^2 Z_2^2 \equiv \frac{1}{3}(f^2 - \eta_1 \overline{f}) \equiv \frac{1}{3}(u_0(\eta_1^2 - 3\eta_3) - \eta_1)\overline{f} \pmod{J} \ , \end{array}$$

we have

$$(1 + u_0 u_2)\eta_3 = \frac{1}{3}(9u_1\eta_2 - u_2\eta_1 + u_0 u_2 \eta_1^2) \ .$$

Using (36), one has

$$\begin{array}{l} u_0 \overline{f}^2 f^2 - \overline{f}^3 \equiv 3(\omega^2 - \omega)\xi \ , \qquad 2u_0(\eta_1^2 - 3\eta_3)^2 - 2\overline{f}^3 \equiv 6(\omega^2 - \omega)\xi \pmod{J}, \\ 2\overline{f}^3 \equiv 27\eta_2^3 - 9\eta_1\eta_3 + 2\eta_1^3 - 3(\omega^2 - \omega)\xi \pmod{J} \ , \end{array}$$

hence

$$3(\omega^2 - \omega)\xi = 2u_0(\eta_1^2 - 3\eta_3)^2 - 27\eta_2^3 + 9\eta_1\eta_3 - 2\eta_1^3 \ .$$

Using the above relations, we have

$$\begin{array}{l} (1 + u_0 u_2)(1 - u_0\eta_1)(Z_1 q_2 - Z_2 q_1 - u_1(\omega - \omega^2) q_3) \\ + \frac{1}{2+\omega}(1 + u_0 u_2)\left[-\omega^2 u_2 Z_3 (f - u_0 \overline{f}^2) + u_0 u_2 Z_3 (f^2 - u_0(\eta_1 - 3\eta_3)\overline{f} \right] \equiv (Z_3 \overline{f} - Z_3 \omega \eta_1)(1 - u_0 \eta_1)(\eta_1 + u_2 + u_0 u_2^2 \end{array}$$

As $Z_3\overline{f}, Z_3$ are two basis elements of $\mathbb{C}[Z]/J$, their coefficients in the last term of the above relation are zero. This implies

$$\eta_1 = -u_2 - u_0 u_2^2 + 3u_0^2 u_3^2 \ .$$

From all the above relations between u_i, η_j, ξ, one concludes that (u_0, u_2, u_3) forms a coordinate system of $\mathrm{Hilb}^G(\mathbb{C}^3)$ centered at x_0 and the following relations hold,

$$\begin{array}{ll} u_1 & = -u_0 u_3, \qquad\qquad \eta_1 = -u_2 - u_0 u_2^2 + 3u_0^2 u_3^2, \\ \eta_2 & = \frac{1}{3} u_3 (1 + u_0 u_2 + u_0^2 u_2^2 - 3u_0^3 u_3^2), \qquad \eta_3 = \frac{1}{3}(u_2^2 - 3u_0 u_3^2)(1 + u_0 u_2 + u_0^2 u_2^2 - 3u_0^3 u_3^2) \\ \xi & = \frac{\omega - \omega^2}{9}(-1 + u_0 u_2)(3u_3^2 + u_2^3 - 3u_0 u_2 u_3^2)(1 + u_0 u_2 + u_0^2 u_2^2 - 3u_0^3 u_3^2). \end{array}$$

Again, the above expressions implies the relation (34), and the G-invariant rational function expression of u_is are given as follows,

$$\begin{array}{llll} u_0 = & \frac{6\eta_3 - 2\eta_1^2}{3(\omega^2 - \omega)\xi + 9\eta_1\eta_3 - 2\eta_1^2 - 27\eta_2^2}, & u_1 = & \frac{\eta_2(-6\eta_3 + 2\eta_1^2)}{(\omega^2 - \omega)\xi + \eta_1\eta_3 - 9\eta_2^2}, \\ u_2 = & \frac{-(\omega^2 - \omega)\xi\eta_1 - 6\eta_3^2 + 9\eta_1\eta_2^2 + \eta_1^2\eta_3}{(\omega^2 - \omega)\xi + \eta_1\eta_3 - 9\eta_2^2}, & u_3 = & \frac{\eta_2(3(\omega^2 - \omega)\xi + 9\eta_1\eta_3 - 27\eta_2^2 - 2\eta_1^3)}{(\omega^2 - \omega)\xi + \eta_1\eta_3 - 9\eta_2^2}, \end{array}$$

hence $dZ_1 \wedge dZ_2 \wedge dZ_3 = \frac{\omega - \omega^2}{12} du_0 \wedge du_2 \wedge du_3$.

For J near the element x_0' in $\mathrm{Hilb}^G(\mathbb{C}^3)$, we have

$$J = \langle \overline{f} - u_0' f^2, \ m_9 - u_1' m_5 - u_2' m_4 - u_3' m_1, \ Y_1 - \eta_1, Y_2 - \eta_2, Y_3 - \eta_3, X - \xi \rangle \ .$$

By a similar argument as the case x_0, one obtains that (u'_0, u'_2, u'_3) is an affine coordinate system with

$$
\begin{array}{ll}
u'_1 & = -u'_0 u'_3, \qquad\qquad \eta_1 = -u'_2 - u'_0 u'^2_2 + 3u'^2_0 u'^2_3, \\
\eta_2 & = \frac{1}{3} u'_3 (1 + u'_0 u'_2 + u'^2_0 u'^2_2 - 3u'^3_0 u'^2_3), \qquad \eta_3 = \frac{1}{3}(u'^2_2 - 3u'_0 u'^2_3)(1 + u'_0 u'_2 + u'^2_0 u'^2_2 - 3u'^3_0 u'^2_3) \\
\xi & = \frac{\omega^2-\omega}{9}(-1 + u'_0 u'_2)(3u'^2_3 + u'^3_2 - 3u'_0 u'_2 u'^2_3)(1 + u'_0 u'_2 + u'^2_0 u'^2_2 - 3u'^3_0 u'^2_3),
\end{array}
$$

and the following relations hold,

$$
\begin{array}{llll}
u'_0 = & \frac{6\eta_3 - 2\eta_1^2}{3(\omega-\omega^2)\xi + 9\eta_1\eta_3 - 2\eta_1^2 - 27\eta_2^2}, & u_1 = & \frac{\eta_2(-6\eta_3 + 2\eta_1^2)}{(\omega-\omega^2)\xi + \eta_1\eta_3 - 9\eta_2^2}, \\
u'_2 = & \frac{-(\omega-\omega^2)\xi\eta_1 - 6\eta_3^2 + 9\eta_1\eta_2^2 + \eta_1^2\eta_3}{(\omega-\omega^2)\xi + \eta_1\eta_3 - 9\eta_2^2}, & u'_3 = & \frac{\eta_2(3(\omega-\omega^2)\xi + 9\eta_1\eta_3 - 27\eta_2^2 - 2\eta_1^3)}{(\omega-\omega^2)\xi + \eta_1\eta_3 - 9\eta_2^2},
\end{array}
$$

hence $dZ_1 \wedge dZ_2 \wedge dZ_3 = \frac{\omega^2-\omega}{12} du'_0 \wedge du'_2 \wedge du'_3$.

With the analysis we have made in this section, one concludes that $\mathrm{Hilb}^G(\mathbb{C}^3)$ is covered by four affine spaces $\mathbb{C}^3$ centered at the central elements in (38), and the G-invariant volume form $dZ_1 \wedge dZ_2 \wedge dZ_3$ of $\mathbb{C}^3$ induces a never-vanishing global volume form of $\mathrm{Hilb}^G(\mathbb{C}^3)$. This completes the proof of Theorem 5.1.

6 Concluding Remarks

In this article, we have provided a detailed derivation of the smooth toric structure of $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$. Its relation with crepant resolutions of $\mathbb{C}^4/A_r(4)$ has been found, and different crepant resolutions connected by flops of 4-folds can be visualized in the process. We have also given a constructive verification of the smooth and crepant properties of $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ by a direct computation method. In the abelian case $A_r(4)$, the solution has been given in Sects. 3, 4 by the standard toric method, a combinatorial mechanism built upon monomials in $\mathbb{C}[Z]$, which can be regarded as characters of the whole torus group T_0, containing $A_r(4)$ as a finite subgroup. The smooth toric structure of $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$ is derived from a procedure, which mainly consists of two steps: first, one obtains a complete list of monomials ideals in $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$ which correspond to the 0-dimensional toric orbits,(see (21) (30) (32)); second, by the Gröbner basis technique and a detailed analysis of the G-regular module property of $\mathbb{C}[Z]/J$ for an ideal J in $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$, one proceeds to identify the toric coordinates from the ideal-generators of J. In this manner, the explicit form of the canonical bundle of $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$ can be determined as a disjoint sum of exceptional divisors, each of which could be blown down to give rise to crepant resolutions of $\mathbb{C}^4/A_r(4)$. These crepant resolutions are connected by a sequence of flops in 4-folds through $\mathrm{Hilb}^{A_r(4)}(\mathbb{C}^4)$. We intend to apply a similar mechanism to the non-abelian case $G = \mathfrak{A}_{n+1}$, but relying only on the data of G-representations in $\mathbb{C}[Z]$, a "big" group like the torus in the abelian case does not exist in the latter case though. In §6, we have made a detailed study on the structure of $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$, which would serve us as a demonstration of the effectiveness of the method even though its crepant smooth conclusion is known by now [2]. We have succeeded to give an explicit verification of the crepant smooth structure of $\mathrm{Hilb}^{\mathfrak{A}_4}(\mathbb{C}^3)$ following our thought by a direct constructive method via group representations. A similar analysis to the higher dimensional cases is now under progress and partial results are promising. As to the role of G-Hilbert scheme in the study of crepant resolution of S_G, our conclusion for the case $G = A_r(4)$ has indicated the non-crepant property of $\mathrm{Hilb}^G(\mathbb{C}^4)$, but with a intimate relation with crepant resolutions of S_G. For higher dimensional case, this kind of link between $\mathrm{Hilb}^G(\mathbb{C}^n)$ and some possibly existing crepant resolutions of S_G could be further loosely related. However, the G-Hilbert scheme would still be worth for further study on its own right due to the built-in character of group representations into the geometry of orbifolds. This could be a promising direction of the geometrical study of singularity. Such program is now under our consideration for the future study.

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